[16] N.C. Wormald, Classifying k-connected cubic graphs, *Lecture Notes in Mathematics; New York, Springer Verlag*, **748**, (1979), 199-206.

zation, Von Humboldt Postdoctoral Fellowships as well as the Center for Discrete Mathematics and Computer Science (DIMACS). Finally a debt of gratitude is owed U.S.R Murty for his significant efforts and contributions during the revision of this paper.

References

[1] B. Alspach, L.A. Goddyn, C.Q. Zhang, Graphs with the circuit cover property, *Trans. Amer. Math. Soc.* (submitted).

[2] K. Appel, W. Haken, Every planar map is four-colourable, part I: discharging, *Illinois Journal of Math.* **21** (1977), 429-490.

[3] K. Appel, W. Haken, J. Koch, Every planar map is four-colourable, part II: reducibility, *Illinois Journal of Math.* **21** (1977), 491-567.

[4] J. Edmonds, Maximum matching and a polyhedron with (0,1) vertices, *J. Res. Nat. Bur. Standards Sect. B*, **69B** (1965), 125-130.

[5] M.N. Ellingham, Petersen subdivisions in some regular graphs, *Congressus Numerantium*, **44** (1984), 33-40.

[6] M. Fontet, Graphes 4-essentiels, *C.R. Acad. Sci. Paris*, **t287** (1978), 289-290.

[7] L.A. Goddyn, Cones, lattices and Hilbert bases of circuits and perfect matchings, to appear in *Contemporary Mathematics.*

[8] I. Holyer, The NP-completeness of edge-colouring, *SIAM J. Computing* **10** (1981), 718-720.

[9] A.K. Kelmans, On 3-connected graphs without essential 3-cuts or triangles, *Soviet Math. Dokl.* **33** (1986) No.3, 698-703.

[10] L. Lovász, Matching structure and the matching lattice, *Journal of Combinatorial Theory (B)* **43** (1987), 187-222.

[11] P.D. Seymour, On multi-colourings of cubic graphs and conjectures of Fulkerson and Tutte, *Proc. London Math. Society*, (3) **38** (1979), 423-460.

[12] P.D. Seymour, On Tutte's extension of the four-colour problem, *Journal of Combinatorial Theory (B)* **31** (1981), 82-94.

[13] P.G. Tait, Remarks on the colouring of maps, *Proc. Royal Soc. Edinburgh, Ser. A* **10** (1880), 729.

[14] W.T. Tutte, On the algebraic theory of graph colourings, *Journal of Combinatorial Theory (B)*, **23** (1966), 15-50.

[15] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* **114** (1937), 570-590.

S and $V-S$ has a single degree two node, then by minimality it is 3-edge-colourable and these colourings can be combined to give a colouring of G. Otherwise without loss of generality, S contains only degree three nodes. Thus since G has $t \neq 1$ degree two nodes, $V-S$ contains exactly t such nodes. Thus the graphs induced by S and $V-S$ have 2 and $t+2$ degree two nodes respectively, and so adding edges in these two graphs between the ends of edges in the cut $\delta(S)$, results in two fractionally 3-edge-colourable graphs. Thus they are 3-edge-colourable and the colourings can be extended to a colouring of G where the edges of $\delta(S)$ receive the same colour. Thus we may assume that the only 2-edge cuts are trivial and in particular the degree two nodes form a stable set. Moreover, it cannot be that $t \neq 3$ and G has a node of degree three adjacent to two degree two nodes u and w. This is because any 3-edge-colouring of the contracted graph $G/\{vw, vu\}$ can be easily transformed to a 3-edge-colouring of G. It follows that G either (i) arises from subdividing once the edges of a matching of a 3-graph or (ii) it has three independent degree two nodes two of which are adjacent to the same node.

Claim. *G cannot have a 3-edge cut $\delta(S)$ such that $|S|$, $|V(G)\backslash S| > 2$.*
Note that this will imply that (ii) cannot occur.
Proof of claim: Otherwise, let $\{xy, x'y', x''y''\} := \delta(S)$, with $\{x, x', x''\} \subseteq S$ and x, x', x'' distinct since G has no proper 2-edge cuts. Note that both S and $V(G)\backslash S$ must contain degree two nodes, for otherwise we may contract S and $V(G)\backslash S$ to obtain two 3-edge-colourable graphs which can subsequently be combined to show G to be 3-edge-colourable. Let $H_1 = (S \cup \{v, w\}, E(G[S]) \cup \{vx, vx', wx'', vw\})$ and $F_1 = (V(G)\backslash(S \cup \{v, w\}), E(G-S) \cup \{vy, vy', wy'', vw\})$. It can be verified that since G does not have non-trivial 2-edge cuts, H_1 and F_1 are $(P_{10}\backslash v)$-free. Moreover, by hypothesis they both have a 3-edge-colouring since they each have at least two degree two nodes. Observe also that in such a colouring of H_1 (respectively F_1), wx'' (respectively wy'') and one of vx, vx' (respectively vy, vy') belong to the same colour class. Let H_2 (respectively F_2) be obtained from H_1 (respectively F_1) by deleting the edge incident to v that belongs to the same colour class as wx'' (respectively wy'') and by joining its other end to w with a new edge. Again, H_2, F_2 are 3-edge-colourable. In addition, the colouring of one of H_1, H_2 can be combined with that of one of F_1, F_2 to yield the required colouring of G, a contradiction.

To conclude, we observe that by Corollary 4.2, G must be planar. Otherwise, the graph obtained from suppressing all degree two nodes of G is a Möbius ladder $M(k)$, $k \leq 5$. Thus, G is obtained from $M(k)$ by subdividing the edges of a matching. Moreover, since G is $(P_{10}\backslash v)$-free, the matching is of size at most two and if $k \geq 4$, then none of its edges can be rungs. But then G is 3-edge-colourable, a contradiction. □

Acknowledgements: This work was completed while the authors were visiting the Research Institute for Discrete Mathematics, Bonn and the Center for Mathematics and Computer Science, Amsterdam. We are grateful for the excellent working environments afforded by these institutions. We also wish to thank Jean Fonlupt, Luis Goddyn, Bruce Reed and the two referees for their stimulating and beneficial comments. The first author acknowledges the support of a postdoctoral fellowship held at the Universitè du Quebéc à Montréal and the second author acknowledges support from North Atlantic Treaty Organi-

by deleting the edge $x_i x_{i+1}$ and by contracting $x_i x_{i-1}$ and $x_{i+1} x_{i+2}$. Moreover, $P_{w_1 w_2}$ is an H'-distant path of G with one attachment edge in a twisted band B' and the other in $\delta(V(B'))$. In addition, $n(H') = n(H)$. However, as shown before, such a configuration contradicts the fact that H' maximizes (p, n) (see Figure 10). □

We now complete the proof of the theorem.

Lemma 4.13 *If H is a local ring minor of G, then $\Gamma(A)$ is a 2-patch and $\Gamma(B)$ is a 1-patch.*

Proof. That $\Gamma(B)$ is an 1-patch follows easily from (4.7) and the fact that H is a local ring. We prove that $\Gamma(A)$ is a 2-patch by induction on $|E(G)|$.

Let $x_0 y_0$ and $x_1 y_1$ be the links of A. If $\Gamma(A) = A$, then there is nothing to prove. So, some pair of opposite edges of A must be long. Assume first that $x_0 y_0$ and $x_1 y_1$ are long. By (4.5), one of $x_0 x_1$ and $y_0 y_1$ must be short. Assume that $x_0 x_1$ is short. For $i = 0, 1$, let v_i be chosen adjacent x_i on the path $\pi(x_i y_i)$. By (4.7) and the fact that H is a local ring, there exists an H-external path P having v_0, v_1 as attachment nodes.

Let G' be the graph $(G - x_0 x_1)/\{v_0 x_0, v_1 x_1\}$ and let H' be the ring minor $((H - x_0 x_1) + v_0 v_1)/\{v_0 x_0, v_1 x_1\}$ of G'. It is easy to see that G' is cyclically 4-edge connected and that H' together with $A' = (v_0, y_0, y_1, v_1)$ and $B' = B$ is a local ring of G'. Thus, by the induction hypothesis, $\Gamma(A')$ is a 2-patch. It follows that $\Gamma(A)$ is a 2-patch. The case where the two non-link edges of A are long is identical. □

5 Applications to Edge Colourings

We now prove the relaxation of Tutte's Conjecture which weakens the forbidden minor restriction.

Theorem 5.1 *If G is a 2-edge connected, cubic graph with no minor isomorphic to the Petersen Graph minus an edge ($P_{10} \backslash e$), then G is 3-edge-colourable.*

Proof: Suppose that G is a minor minimal counterexample to the statement. Then G must not be reducible, or we could colour the two "shrunken" graphs and extend back to a colouring for G. If G is planar, then its 3-edge-colourability would follow from the four node-colourability of a planar graph. Thus by Theorem 4.6 G is either a Möbius Ladder or a patched ring and in either case G has a 4-circuit $C = (v_1, v_2, v_3, v_4)$. Consider the graph $G' = (G - V(C)) \cup \{w_1 w_2, w_3 w_4\}$, where w_i is the node of $G - V(C)$ adjacent to v_i, $1 \leq i \leq 4$. G' is $(P_{10} \backslash e)$-free and from a 3-edge-colouring of G' we can obtain a 3-edge-colouring of G by assigning the colour of $w_1 w_2$ to $v_1 w_1$, $v_2 w_2$ and the colour of $w_3 w_4$ to $v_3 w_3$, $v_4 w_4$ and then extend this colouring to the edges of C. This completes the proof of the theorem. □

Theorem 5.2 *Any minimal counterexample to Conjecture 2.2 is planar.*

Proof: Suppose that G is a minimal counterexample to the $(1, 2, 3)$ Conjecture. It is straightforward to deduce from Theorem (1.1) that G has maximum degree 3 and can have no cut edge so assume that G is 2-edge connected. Suppose next that $S \subseteq V$ such that $1 < |S| < |V| - 1$ and $|\delta(S)| = 2$. If neither of the two graphs obtained by contracting apply Lemma 3.1 to find a new minor H' whose nodes, edges and expansions of edges outside $\delta_H(y)$ are identical to H, but which satisfies Lemma 3.1(i) or (ii) (recall H has no triangles). The 4-cut property of Claim 4.12 together with the fact that $V(H') = V(H)$ implies that H' again maximizes (p, n, m, k) (and A^*, B^* are unchanged). Hence by Claim 4.11, there can be no H'-distant path with an attachment edge incident to y. It follows that there are two paths which cross on an edge incident to y; there are four possible cases (see Figure 13). In the cases of Figure 13(a) and (c) one easily finds a minor contradicting (4.7)(ii). In the case (b), one may chose a new minor H'' with $(p(H), n(H), m(H)) = (p(H''), n(H''), m(H''))$ and $k(H'') = k(H) + 1$, a contradiction. In the case (d), one creates a new minor H'' by altering the expansion e to use P. Once again H'' maximizes (p, n, m, k) but now there is an H''-distant path which violates Claim 4.11, a final contradiction. Thus we may assume that any H-external path has both attachment edges in B^* or in A^* and so H satisfies properties (i) and (ii) of a local ring.

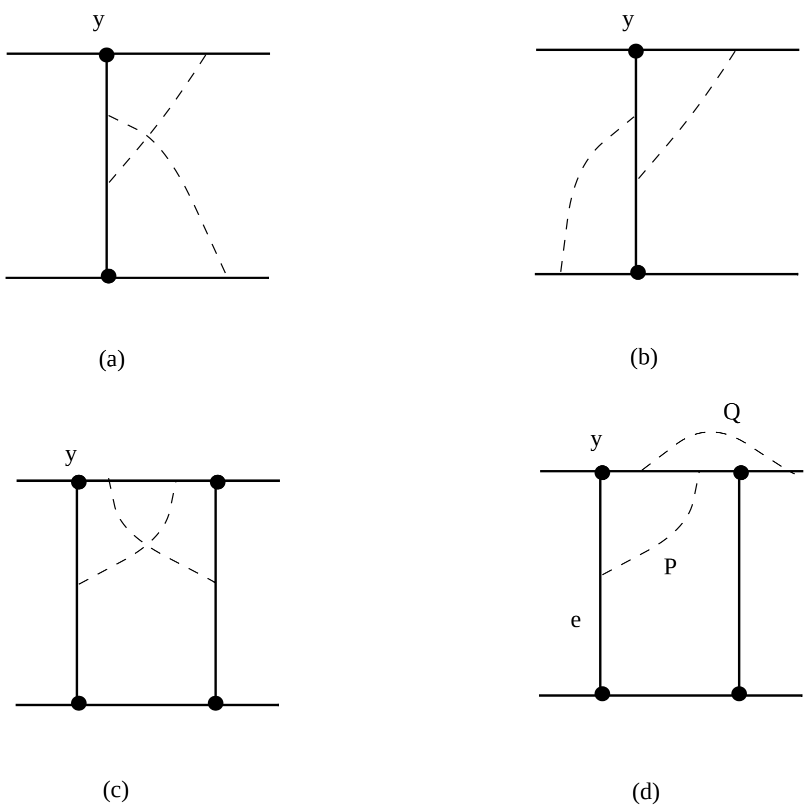


Figure 13:

Suppose now that H does not satisfy property (iii) of a local ring. Thus, there exist H-external paths $P_{v_1v_2}$, $P_{w_1w_2}$ such that $\pi(B) \cup P_{v_1v_2} \cup P_{w_1w_2}$ is a subdivision of K_4 where $B = B^*$ or A^*. By the fact that $p(H)$ is maximized, B is not a simple band, so $B = B^*$. By (4.7), B^* is the unique twisted band of H. Let x_iy_{i+1}, $x_{i+1}y_i$, be the links of B^* and note that x_i, v_1, w_1, y_{i+1} and x_{i+1}, w_2, v_2, y_i appear in this order on $\pi(x_iy_{i+1})$ and $\pi(x_{i+1}y_i)$ respectively. Observe now that G has a minor $H' \cong H$ which is obtained from $H + v_1v_2$

Finally, the case in which v_1 and v_2 belong to two consecutive simple links can be easily excluded with the help of (4.5).

Proof of Lemma 4.9.
Assume that subject to maximizing (p, n, m, k) we have chosen H so that the number of edges in expansions of the four edges of $\delta_H(V(A^*))$ is minimized. We show that H satisfies properties (i) and (ii) of a local ring. We first show that the node sets of A^* and B^* each 'separate' $\Gamma(A^*), \Gamma(B^*)$ from the rest of the graph. This gives us great freedom to choose other minors isomorphic to H without 'disturbing' $\Gamma(A^*), \Gamma(B^*)$ and hence maintaining maximality of (p, n, m, k).

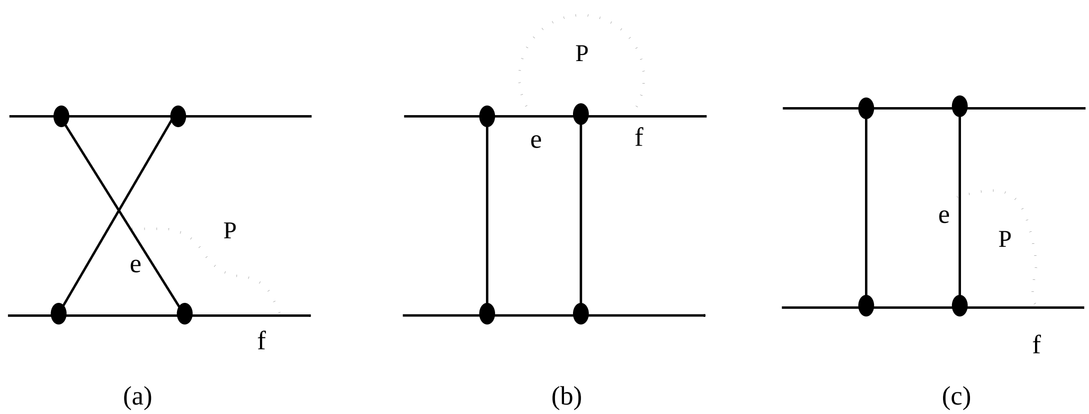


Figure 12:

Claim 4.12 *For any twisted band B we have (i) there is no H-external path with one endpoint in $V(\pi(B))$ and the other in $\pi(H)\backslash V(B)$. We also have (ii) there is no H-external path with one endpoint in $V(\pi(A^*))$ and the other in $\pi(H)\backslash V(A^*)$.*

Proof: Suppose that (i) is not the case. Since the non-link edges of B are short, there exists an H-external path P with attachment edges e, f such that e is a link edge of B and f is not contained in B. Moreover, by Claim 4.11, f must be incident to e (see Figure 12(a)). But then altering H so that the expansion of e uses P, we find a ring minor violating (4.7)(i). So suppose that (ii) is violated and let P be an H-external path with attachement edges e, f such that $e \in E(A^*)$ and $f \notin E(A^*)$. By Claim 4.11, e, f must be incident. There are two possibilities (see Figure 12(b) and (c)). There also can be no H-external path which joins an internal node of P with an internal node of some link join of A^* (or B^*), for otherwise there would be an H-distant path violating Claim 4.11. Thus it follows that in both cases, altering H to use P in its expansion of e produces a new minor which is again maximal. But this minor contradicts the minimality of expansions of edges with one endpoint in $V(A^*)$. □

Now suppose that there is an H-external path P which does not have both attachment edges e, f in A^* or in B^*. Then by Claim 4.11, e, f are incident at some node y, and by Claim 4.12, neither e nor f is in a twisted band or in A^*, and hence neither is y. Without loss of generality, e is an edge of C_1 and f is either in C_1, or is a simple link. In either case we

$x_i x_{j+1}$ is a link join for H''s twisted band but not of B^* in H, and any edge in a link join of B^* in H is necessarily in a link join of H''s twisted band.

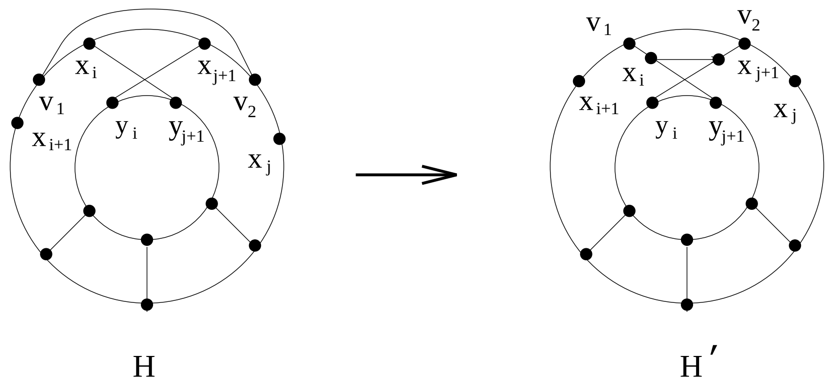


Figure 11:

Further analysis of this subcase depends on where A^* lies with respect to the segment L on C_1 which does not contain an edge of B^*. Without loss of generality, x_{i+1} and x_j lie on L. We let H' denote the obvious ring minor obtained by taking $C_1' := (v_1, v_2, x_{j+1}, \cdots, x_i)$ and C_2' determined by the neighbours of $x_j, x_{j+1}, \ldots, x_i, x_{i+1}$ (i.e., all nodes of C_2 except $y_{i+2}, \ldots, y_{j-1}$). We claim that H' is such that $p(H') = p(H)$, whereas $n(H') > n(H)$. Indeed $p(H') = p(H)$ since no twisted band contains a node of L. To see this, suppose that $\tilde{B}$ is a twisted band containing a node of L. Let L' be the path internally-disjoint from L such that $C_1 = L \cup L'$. Since $v_1, v_2 \notin V(\tilde{B})$ and each non-link edge of $\tilde{B}$ is short, L would contain an edge of $\tilde{B}$. Now one of L, L', say L, then contains the end of some other link which is contained in neither of $B^*, \tilde{B}$. But then the operation of Figure 11 yields a minor contradicting (4.7)(ii).

If A^* is not defined, then by (4.10), each of the two segments of L, L' **must** contain endpoints of the non-link edges of a twisted band. Moreover, one of these segments contains the endpoint of at least one more link (simple or not) of H. Thus the operation of Figure 11 can again be applied to obtain a minor contradicting (4.7)(ii). This together with the argument of Case (i) shows that if A^* is not defined, then H has no distant paths and so $G = H$ and hence G satisfies the hypothesis.

If $|V(L) \cap V(A^*)| < 2$, we claim that $\lambda(A^*) = 0$. For if there is a link join of A^* in H, then both of A^*'s links are long in H and one of these is also a long link in H'. But also v_1, v_2 are incident to long links in H' and so (4.5) implies that H' is not $P_{10}\backslash e$-free. Hence $\lambda(A^*) = 0$. Since the simple band $\tilde{A}^* = (v_1, y_{i+2}, y_{j-1}, v_2)$ has a link join in H', we have $n(H') > n(H)$, a contradiction. So suppose $|V(L) \cap V(A^*)| = 2$. We first show that no edge of $P_{v_1 v_2}$ was in a link join of A^*. If this were not the case, then there would be an H-distant path which violated (4.8) with one attachment being a link of A^*. It now follows that each edge in a link join of A^* in H is also in a link join of the simple band $\tilde{A}^*$ in H'. But in addition, the edges of $\pi_H((x_j, x_{j-1}, \ldots, x_{i+1}))$ are included in a link join of $\tilde{A}^*$. Thus again $n(H') > n(H)$, which is a contradiction.

Case (iv)

When both v_1, v_2 belong to $\pi(A^*)$, by (4.5), at most one of the new simple bands of H' can have a link join. Thus, $n(H') = n(H)$. Also, $(p(H'), m(H')) = (p(H), m(H))$, whereas $k(H') > k(H)$.

Case (ii)
Note that by (4.7)(ii), $p(H) = 1$. Now let H' be the ring minor of G obtained from H as shown in Figure 10. Observe that $p(H) = p(H') = 1$. Clearly, the nodes of A^* are properly contained in the circuit indicated by the solid lines in the drawing of H. In H', x_2x_{k-1} and y_2y_{k-1} are neighbouring simple links, and the band A' induced by them has one of H's simple links as a link join which was not such a join of A^*. In addition, any edge in a link join of A^* is also in one for A'. To see this, note first that the only edges which are H-external but not H'-external are those in $P_{v_1v_2}$ and these could not have been in any link join of A^* by (4.7)(iii). Next note that one of A^*'s link edges is of the form x_iy_i where $2 < i < k-1$; hence any link join of A^* necessarily is a join of A' with at least three attachments of A', two of which are its links. Thus $\lambda(A') > \lambda(A^*)$ and so $n(H') > n(H)$, a contradiction.

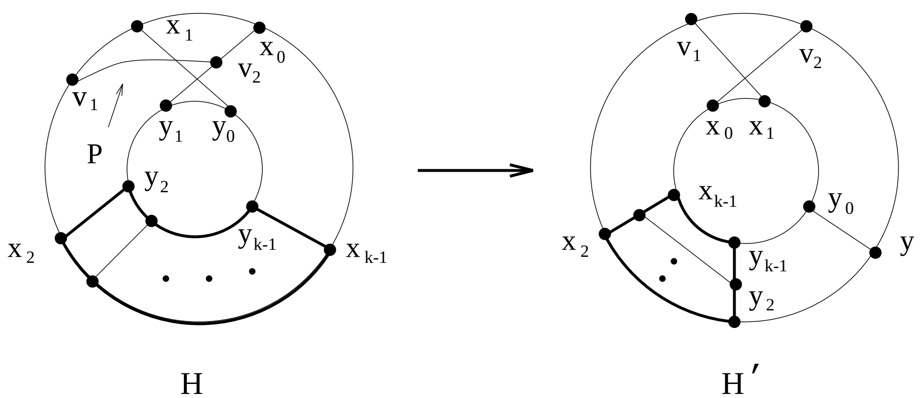


Figure 10:

Case (iii)
The reader may proceed assuming that A^* is defined; the case where it is not will be dealt with explicitly and separately within the proof.
Assume that both v_1, v_2 are on the expansion of C_1, say $v_1 \in V(\pi(x_ix_{i+1}))$ and $v_2 \in V(\pi(x_jx_{j+1}))$. By (4.7)(i),

(2) neither x_ix_{i+1} nor x_jx_{j+1} belongs to a twisted band of H.

Consider the two segments of the circuit $\pi(C_1)$ subtended by v_1v_2. We first consider the subcase in which one of these segments has the property that the only nodes of H on it are the nodes of B^* on C_1. Without loss of generality, let this segment be the one that contains x_{j+1} and x_i. Note that $j+2 = i \pmod k$ and $x_{j+1}x_i$ is the edge of B^* in C_1. By considering the circuits $C_1' := (v_1, x_{i+1}, \cdots, x_j, v_2)$ and $C_2' := C_2$ we obtain a new ring minor H' (see Figure 11). First note that $p(H') = p(H)$. Next note that by (4.7)(iii) no link join of A^* in H has an edge in common with the graph which is the expansion in H' of the twisted band (v_1, y_{j+1}, y_j, v_2). Thus $n(H') = n(H)$. Finally, $m(H') > m(H)$ since where $\geq_L$ is the lexicographic measure and hence gives rise to a total order on the collection of ring minors.

Lemma 4.9 *Any ring minor of G that maximizes (p, n, m, k) is a local ring.*

Let H be a ring minor of G that maximizes (p, n, m, k). Let B^* be a twisted band such that $\lambda(B^*) = m(H)$. If H has a simple band, then let A^* be a simple band such that $\lambda(A^*) = n(H)$; if H does not, then A^* is not defined. We note at this point that:

4.10 *If A^* is not defined, then $p \geq 2$ and H has no consecutive simple links. Also, by (4.7), each non-simple link is short and so by (4.8), any H-distant path has both attachment nodes on $V(\pi(C_1) \cup \pi(C_2))$.*

We shall derive Lemma 4.9 as a consequence of the next claim. Only two of the cases ((i) and (iii)) in the proof of the claim apply to the situation where A^* is not defined, and together with the preceding remarks, one then deduces that in this case, G itself must be a twisted ring. Hence we ignore this case subsequent to the proof of the claim.

Claim 4.11 *The attachment edges of any H-distant path of G are the links of either A^* (if defined) or B^*.*

Proof: Let $P_{v_1 v_2}$ be an H-distant path of G for which the claim does not hold. Then by (4.7) and (4.8) one may assume that one of the following holds:

(i) $v_1 \in V(\pi(C_1))$ and $v_2 \in V(\pi(C_2))$, or

(ii) $v_1 \in \pi(\delta(V(B^*)))$ and $v_2 \in V(\pi(B^*))$, or

(iii) $\{v_1, v_2\} \subset V(\pi(C_1))$ (or $V(\pi(C_2))$), or

(iv) $v_1 \in V(\pi(x_i y_i))$ and $v_2 \in V(\pi(x_{i+1} y_{i+1}))$.

We shall show that none of these possibilities can occur by deriving a contradiction to the fact that H maximizes (p, n, m, k).

Case (i)

Let H' be obtained from H by subdividing the edges whose expansions contain v_1, v_2 and joining the two new degree two nodes. When not both v_1, v_2 are in $V(\pi(A^*))$, it is easy to see that either

- H' is a ring minor with $(p(H'), n(H'), m(H')) = (p(H), n(H), m(H))$ and $k(H') > k(H)$, or
- H' is a ring minor with $p(H') > p(H)$, or
- H' is a minor which has an $\Omega(1, 5)$ with a long non-link edge in a twisted band, and hence has a $(P_{10} \backslash e)$-minor.

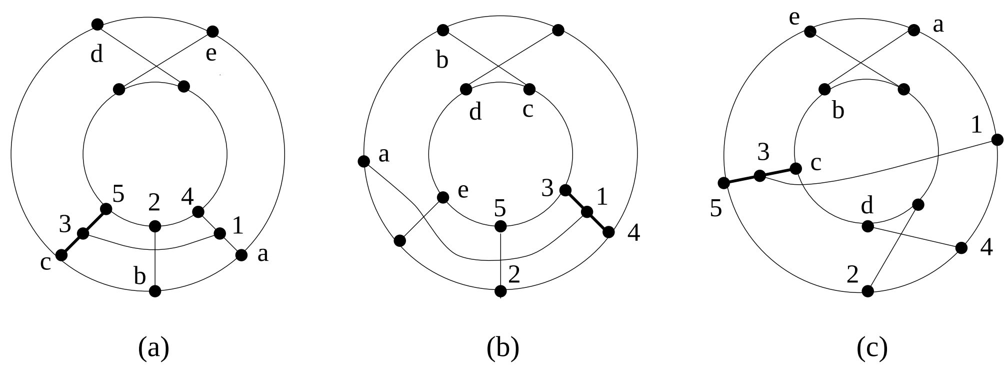


Figure 9:

only one attachment node on a link edge. Moreover, when B is a twisted band of H and P, P' are H-distant paths with attachments edges in $E(B)$ such that $\pi(B) \cup P \cup P'$ is a subdivision of K_4, then B is clearly not a patch. Thus, it is desirable to find a twisted ring minor which does not have such distant paths. Towards this end, we introduce a special type of a ring minor.

Let H be a ring minor of G and let D be a 4-circuit of H. A *join* of D is either a distant path or a 3-bridge of G, relative to H, that has all its attachment edges in D. We denote by $\Gamma(D)$ the subgraph of G induced by $E(\pi(D))$ together with the edges in all joins of D. H is a *local ring* if it has a simple band A and a twisted band B such that the following hold:

(i) The edges of $H - E(A \cup B)$ are short.

(ii) $\Gamma(A) \cap \Gamma(B) = \emptyset$, and every edge of G not in $\pi(H - E(A \cup B))$ belongs either to $\Gamma(A)$ or to $\Gamma(B)$.

(iii) For any two H-distant paths P, P' of G, $\pi(B) \cup P \cup P'$ is not a subdivision of K_4.

If a given ring minor is not local, then we can perform certain transformations which result in a ring minor which is 'more' local. For formulating the required notion of optimality, we shall need to attach certain parameters with any given ring minor H. Unfortunately, simple parameters such as $k(H)$ and $p(H)$ do not suffice. For this reason, we need to take into account the local structure of H. This is the motivation for the definitions below.

A join of a 4-circuit D whose attachment edges include the link edges of D is called a *link join*. We denote by $\lambda(D)$ the number of edges in all link joins of D. Let $\mathcal{A}$ (respectively $\mathcal{B}$) be the family of simple bands (respectively twisted bands) of H. Let

$$n(H) = \max\{\lambda(A) : A \in \mathcal{A}\}$$

and let

$$m(H) = \max\{\lambda(B) : B \in \mathcal{B}\}.$$

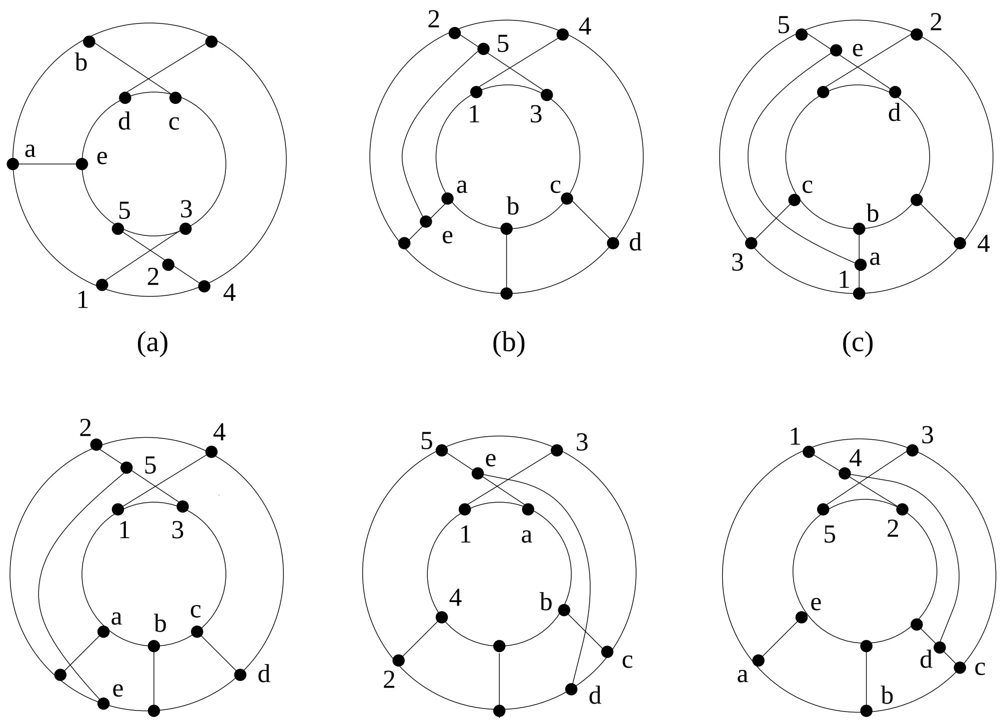
If $\mathcal{A} = \emptyset$, then we set $n(H) = 0$. We say that H maximizes (p, n, m, k), if for every ring minor H' of G,

$$(p(H), n(H), m(H), k(H)) \geq_L (p(H'), n(H'), m(H'), k(H'))$$

Proof. (i). It is immediate from observing the third graph of Figure 7.

(ii). One verifies that the labelling of Figure 8(a) indicates that this graph has $P_{10}\backslash e$ as a minor.

(iii). By (i) and (ii), if P has an attachment edge in B, then it must be a link of H and $p = 1$. Figures 8(b), (c), (d), (e) and (f) exclude the possibility that the other attachment edge is not in $E(B) \cup \delta(V(B))$. $\square$



(a) (b) (c)

(d) (e) (f)

Figure 8:

We now consider H-distant paths having a simple link as an attachment edge.

4.8 *Assume f and g are the attachment edges of an H-distant path. If f is a simple link, then g is also a simple link and f, g induce a simple band.*

Proof: By (4.7), it cannot be that g belongs to a twisted band of H. Also, Figure 9(a) excludes the possibility of g being a simple link and f, g do not induce a simple band of H. Finally, when g is an edge of C_1 or C_2, by (4.7)(i) and (ii), the only cases to consider are the ones excluded in Figure 9(b) and (c). (The other cases are also ruled out by Figure 8.) $\square$

From the above observations we can see the desired ring structure evolving. However, these observations do not rule out the possibility that a given ring minor H may have distant paths with both attachment edges on the same C_i, nor do they rule out distant paths with now finds $P_{10}\backslash v$ in G by considering the union of three sets of paths: (i) the three paths just defined, (ii) the three paths in G_2 from v' to $N_{G_2}(v_2)$ (each of which has length at least two since they each contain an x_i) and (iii) three paths in G which emanate from a node of G_1 and terminate in $N_{G_2}(v_2)$.

To prove sufficiency it is enough to check that the following hold for each extraction of a planar 3-graph:
(i) deleting an arbitrary edge incident to each fork node results in a planar graph, and
(ii) if H is a $K_{3,3}$ minor with fork node $v \in V(H)$, then $\delta_H(v)$ consists of only short edges. □

4.2 The Petersen Graph minus an Edge

We now turn to the proof of Theorem 4.6. Before proving the theorem, we note first that $R_1 \cong \Omega(1,5)$ and $R_2 \cong \Omega(2,5)$. We also remark that the nodes of the Petersen graph can be labelled so that its edges can be partitioned into two 5-circuits (a,b,c,d,e), $(1,3,5,2,4)$ and the edges $a1$, $b2$, $c3$, $d4$, $e5$. Thus if we delete the edge joining the two unlabelled nodes of the graph in Figure 9(a), we obtain a subdivision of the Petersen Graph. We have the following lemma which is used repeatedly in analyzing the structure of $(P_{10}\backslash e)$-free graphs.

4.5 *Let x_iy_i, $1 \leq i \leq 3$ be the simple links of an $\Omega(1,5)$ ring minor H. Let $Q = (H + v_1v_3)$, where v_1, v_3 are nodes that subdivide x_1y_1, x_3y_3 respectively. Then, neither $Q - v_1v_3$ nor $Q - x_2y_2$ is $(P_{10}\backslash e)$-free.*

Theorem 4.6 *Every cyclically 4-edge connected, cubic, $(P_{10}\backslash e)$-free graph G is either planar, a Möbius ladder or a patched ring.*

Proof. If G is neither planar nor a Möbius ladder, then by Theorem 4.1, it has a prayer as a minor. Moreover, since G is $(P_{10}\backslash e)$-free, it has a twisted ring as a minor. We shall show that we can choose such a ring minor H with two 4-circuits, a simple and a twisted band, which "contain" all H-external paths of G. That is, each H-external path of G has both its attachment edges on such a circuit. Subsequently, we show that G arises from simply patching these circuits of H.

Let H be an $\Omega(p,k)$ minor of G. We begin by making a few preliminary observations regarding the attachment edges of H-external paths of G. These observations amount to saying that certain types of external paths cannot exist relative to any ring minor based on the hypothesis that G is $(P_{10}\backslash e)$-free.

First we consider H-external paths of G having an attachment edge in a twisted band of H.

4.7 *Let B be a twisted band of H. Then*

(i) *the non-link elements of B are short,*

(ii) *the link elements of B are short when $p(H) > 1$, and*

(iii) *any H-distant path P that has an attachment edge in B, has both its attachment edges in $E(B) \cup \delta(V(B))$.*

contains a minor isomorphic to F, a contradiction. It follows that 1 and 3-sums maintain the absence of prayers.

So we now suppose that G is a minimal counterexample to the "only if" part. Clearly G is reducible, otherwise by Theorem 4.1, G is either planar or a Möbius Ladder. If G has a cut edge, then deleting this edge and suppressing the degree two nodes produces minors H_1, H_2 of G which are prayer-free. Thus by minimality each H_i satisfies the statement and hence so does G since it is a 1-sum of H_1 and H_2. So assume G has a 2 or 3-edge cut. Note that if G has a 2-edge cut, then either it also has a proper 3-edge cut or G is obtained by doubling a pair of independent edges in a 4-circuit, in which case G is planar. So we assume without loss of generality that G is connected and has a proper 3-edge cut $\delta(S)$. Let G_1, G_2 be the graphs obtained by contracting S and $V - S$ respectively to a single node. Since, these graphs are minors of G, they are prayer-free and so are obtained by repeated 1 and 3-sums as desired. But clearly then G is a 3-sum of G_1, G_2 relative to their corresponding "shrunken" nodes. □

For any positive integer k, and a cubic graph G with a node v, a *k-extension* on v is the operation where each of the edges e_1, e_2, e_3 incident to v is replaced by a path $P_i = (v, x_{i1}, x_{i2}, \ldots, x_{ik}, u_i)$, where $e_i = vu_i$, $i = 1, 2, 3$. In addition, k new *fork nodes* $f_1, f_2, \ldots, f_k$ are created so that f_j is adjacent to x_{1j}, x_{2j}, x_{3j}. An *extraction* of G is a graph obtained by performing extensions on some subset $S \subseteq V$.

Corollary 4.4 *A 3-graph is $(P_{10}\backslash v)$-free if and only if it is $M(4), M(5)$ or an extraction of a planar 3-graph.*

Proof: Suppose that G is a $(P_{10}\backslash v)$-free 3-graph which is a minimum counterexample to necessity. We remark first that for $k \geq 4$, subdividing a rung of $M(k)$ produces a graph with a $P_{10}\backslash v$ minor and so a 3-sum of such a ladder with any 3-graph is not $P_{10}\backslash v$-free. Thus by Corollary 4.3,

(1) G is obtained by repeated 3-sums starting from planar graphs and $K_{3,3}$'s.

By Corollary 4.2, we may assume that G is not cyclically 4-edge connected and so it is the 3-sum of two 3-graphs G_1, G_2 at nodes v_1, v_2. If G_1, G_2 are planar, then so is G. So without loss of generality, G_1 is nonplanar. Moreover, assume that there is no 3-sum resulting in G which has a nonplanar part with fewer than $|V(G_1)|$ nodes. It follows easily that G_1 is cyclically 4-edge connected and hence by (1) $G_1 \cong K_{3,3}$. Note that it is straightforward that G is obtained by a single 1-extension of v_2 in G_2. By minimality of G and (1), G_2 is the extraction of some 3-graph $\tilde{G}_2$ which is planar. If v_2 was not a fork node of G_2, then clearly G is also an extraction of $\tilde{G}_2$, so suppose that v_2 was a fork node in an extension of some v' in $\tilde{G}_2$. Let H_1, H_2 be the graphs induced by the components of $G_2 - N_{G_2}(v_2)$ where $v' \in V(H_1)$. If H_1 consists of a single node, then one deduces that G is isomorphic to the graph obtained from $\tilde{G}_2$ by the extensions applied to derive G_2 except that if v' was k-extended in obtaining G_2, then it is $k+1$-extended to obtain G.

Thus we suppose that H_1 contains the neighbours x_1, x_2, x_3 of some other fork node of G_2. Note first that if $|V(H_2)| = 1$, then G is obtained by performing extensions on a single node of a cubic planar graph consisting of two nodes and three parallel edges. If this is not the case, then $G_2[V(H_2) \cup N_{G_2}(v_2)]$ contains three internally-disjoint paths from a node x in H_2 to the set $N_{G_2}(v_2)$ such that at least one of these paths has length at least two. One

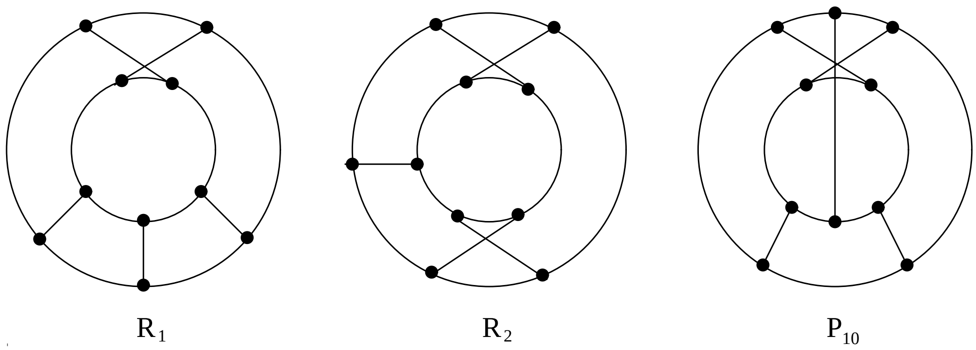


Figure 7: The three prayers

edges on the expansions of two independent edges of H. Since k is assumed to be maximum, it cannot be that these edges are $x_i x_{i+1}$ and $x_{i+k}x_{i+k+1}$, $0 \leq i \leq k-1$. Hence, there must be two distinct 4-circuits in H, C_1 and C_2, on which P has attachment edges. If these two circuits can be chosen so that $V(C_1) \cap V(C_2) = \emptyset$, then there are in H distinct paths P_j, $1 \leq j \leq 4$, with ends in C_1 and C_2. In addition, these paths together with C_1 and C_2 induce a non-planar subgraph of H. This immediately implies that $(\cup_{j=1}^4 P_j) \cup (\cup_{i=1}^2 \pi(C_i)) \cup P$ contains a prayer as a minor, a contradiction. Similarly, if $V(C_1) \cap V(C_2) \neq \emptyset$, then, without loss of generality, the ends, say v and w, of P lie respectively on $\pi(x_i x_{i+1})$ and $\pi(x_{i+k+1}x_{i+k+2})$, for some $i \leq k-1$. In this case, contrary to our hypothesis, the expansions of the 5-circuits of an R_1 minor of G can be taken to be the expansions of the circuits $(x_0, \cdots, x_i, v, w, x_{i+k+2}, \cdots, x_{i+2k-1})$ and $(x_k, \cdots, x_{i+k+1}, x_{i+1}, \cdots, x_{k-1})$ of $H + vw$. The proof is complete. □

It can be checked that $M(k)$ is $(P_{10}\backslash v)$-free if and only if $k \leq 5$. Also, each prayer contains $P_{10}\backslash v$ as a minor. Thus,

Corollary 4.2 *A cyclically 4-edge connected cubic graph is $(P_{10}\backslash v)$-free if and only if it is either planar or a Möbius k-ladder, $k \leq 5$.*

Theorem 4.1 also leads to a full decomposition theorem. A 3-*sum* of two cubic graphs G, H at nodes $v \in V(G), w \in V(H)$ is any graph obtained by deleting v, w in each of the graphs and adding a 3-matching between the sets $N_G(v)$ and $N_H(w)$. A 1-*sum* is a graph obtained by subdividing an edge in each of the graphs and joining the resulting degree two nodes.

Corollary 4.3 *A cubic graph is prayer-free if and only if it is obtained from cubic planar graphs and Möbius Ladders by repeated* 1 *and* 3*-sums.*

Proof: We show first the "if" part and so suppose that G, H are disjoint cubic prayer-free graphs. It is clear that any 1-sum of these graphs is again prayer-free so we consider only 3-sums. Let $v \in V(G)$, $w \in V(H)$ and let G^* be any 3-sum obtained by adding three edges between $N_G(v)$ and $N_H(w)$ in the graph $(G - \{v\}) \cup (H - \{w\})$. Now if G^* contains a prayer minor F, then since each prayer is cyclically 4-edge connected, either $|V(F) \cap V(H - \{w\})| \leq 1$ or $|V(F) \cap V(G - \{v\})| \leq 1$; assume the latter. Hence H itself

When the latter configuration occurs, let v_1, u_1 and w_1 be respectively the other neighbours of v, u and w. Since $T \not\cong K_4$, we may assume with no loss of generality, that $u_1 \neq v_1$. We now choose H and the required H-distant path as follows. H has node set $(V(T)\backslash\{u,v\}) \cup \{y_2, y_3\}$ and in addition, $\pi_H(y_2w) := \pi_T(vw)[y_2, w]$, $\pi_H(y_2v_1) := \pi_T(vw)[v, y_2] \cup \pi_T(vv_1)$, $\pi_H(y_2y_3) := P_{yy_2} \cup P_{yy_3}$, $\pi_H(y_3w) := \pi_T(uw)[y_3, w]$, and $\pi_H(y_3u_1) := \pi_T(uw)[u, y_3] \cup \pi_T(uu_1)$. And for each other edge e, $\pi_H(e) := \pi_T(e)$. Finally $\pi_T(uv)$ is the required H-distant path. □

The above results are related to, and can be used to prove, one due to Fontet [6] and independently to Wormald [16].

Theorem 3.4 (Fontet, Wormald) *A cubic graph different from K_4 is cyclically 4-edge connected if and only if it can be constructed from the cube by repeated edge extensions.*

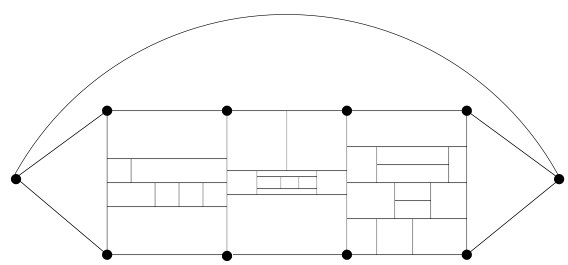


Figure 6:

(An *edge extension* of the graph G relative to two independent edges, is the operation whereby the edges are subdivided and the new degree two nodes are joined). We mention that Corollary 3.3 is best possible in the sense that there exist 3-graphs $G \neq K_4$ with a K_4 minor but no distant H-external paths. These can be constructed from a prism by edge extensions, see Figure 6.

4 The Decompositions

4.1 Graphs without a Prayer

A nonplanar 3-graph is a *prayer* if its nodes can be partitioned into two chordless 5-circuits. Figure 7 displays the three prayers (including the Petersen graph P_{10}).

Theorem 4.1 *A cyclically 4-edge connected cubic graph G is prayer-free if and only if it is planar or a Möbius ladder.*

Proof: Planar graphs are by definition prayer-free. Also, any circuit C of a Möbius k-ladder has the property that either $|\delta(V(C))| = 4$, or $M(k)\backslash V(C)$ contains no circuit. This implies that a Möbius k-ladder does not contain a prayer as a minor.

Conversely, let G be a prayer-free graph and let H be an $M(k)$ minor of G for maximum k. We show, by contradiction, that $H \cong G$. By Theorem 2.1, $k \geq 3$. By Corollary 3.3, if $k = 3$ then $G \cong H$ for any edge extension of $K_{3,3}$ yields a graph isomorphic to $M(4)$. Otherwise, by Corollary 3.3, H can be selected so that G has an H-distant path P with attachment

Proof: If G does not have a T-distant path, then by Lemma 3.1 we may assume that one of the following configurations must have occurred. Either there are T-external paths $P_{v_1v_2}$, $P_{w_1w_2}$ crossing on an edge xy of T (see Figure 4), or G has distinct nodes y, $y_i \notin V(T)$, $1 \leq i \leq 3$, and internally disjoint paths P_{yy_i} such that, $V(\pi(T)) \cap V(P_{yy_i}) = \{y_i\}$, $1 \leq i \leq 3$, and y_1, y_2, y_3 lie on the expansions of the three different sides of a triangle (u, v, w) of T (see Figure 5). We examine these cases below.

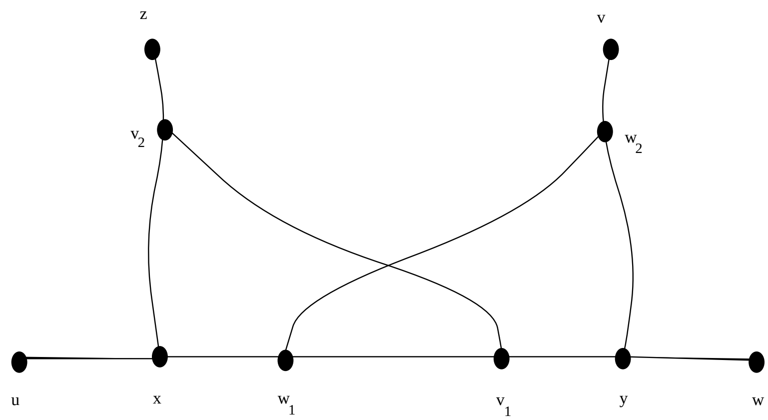


Figure 4:

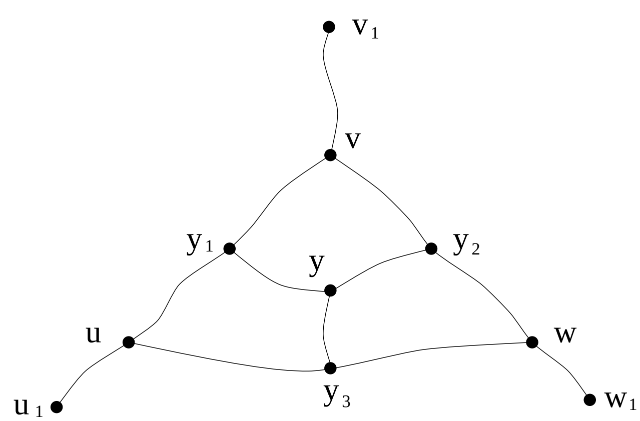


Figure 5:

Suppose first that the former configuration has occurred. Let z and u be the other neighbours of x where v_2 lies on $\pi(xz)$. We attempt to choose H and the required H-distant path as follows: H has the same node set as T except that the node x is changed to node v_1. The expansions of the edges of H are chosen as follows. $\pi_H(v_1y) := \pi_T(xy)[y, v_1]$, $\pi_H(v_1z) := \pi_T(xz)[z, v_2] \cup P_{v_1v_2}$, $\pi_H(v_1u) := \pi_T(xy)[v_1, x] \cup \pi_T(xu)$ and the expansion of any other edge of H is the same as that of its image in T. Finally, $P_{w_1w_2}$ is the distant H-external path specified by the corollary unless $u = v$. If $u = v$, then we can choose a minor H' with the configuration of Figure 5 by changing node x to v_2 and keeping the expansions of H''s edges as in H except for those in $\delta_{H'}(v_2)$ which become $(\pi_T(\delta_T(x)) \cup P_{v_1v_2}) - \pi_T(xy)[v_1, x]$. The nodes y, y_1, y_2, y_3 can be chosen as w_1, x, v_1, w_2 respectively.

If $y \in V(T)$ is incident to the attachment edges of some T-external path, then we can choose the external subgraphs described in (i), (ii), (iii) so that in (i) the distant path has an attachment edge in $\delta_H(y)$; in (ii), the two paths cross on an edge in $\delta_H(y)$; and in (iii) two of the edges of the triangle are incident to y. Moreover, we can choose H so that $V(T) = V(H)$, $E(T) = E(H)$ and for each $e \notin \delta_H(y)$, $\pi_H(e) = \pi_T(e)$.

Proof: Suppose that T itself is not the required minor. Thus, since G is 3-edge connected and $T \not\cong G$, it has a node $y \in V(T)$ and an T-external path whose attachment edges are distinct elements of $\delta_H(y)$. Let x_1, x_2, x_3 be the neighbours of y in T. We begin by defining a family $\mathcal{H}$ of minors of G from which we shall select the one that satisfies the lemma.

A minor H of G belongs to $\mathcal{H}$ if and only if $H \cong T$, $V(H) = V(T)$, $E(H) = E(T)$ and for each $e \in E(H)$, $e \notin \delta_H(y)$, $\pi_H(e) = \pi_T(e)$. Observe that because G is cubic, for $i = 1, 2, 3$, the successor of x_i on $\pi_H(x_i y)$ is the same as the successor of x_i on $\pi_T(x_i y)$. Thus,

3.2 *For any $H \in \mathcal{H}$, y is incident to two edges of H which are the attachment edges of some H-external path of G.*

For each $H \in \mathcal{H}$, and each $i = 1, 2, 3$ we choose $v_i \in \pi_H(yx_i)$ as follows. If there is no H-external path P attached respectively to an internal node of $\pi_H(yx_i)$ and some $\pi_H(yx_j)$, for $j \neq i$, then $v_i = y$. Otherwise, we choose an attachment node v_i of such a path that is as close as possible to x_i. For the minor H we now define $k_i(H) = |E(\pi_H(x_i y)[x_i, v_i])|$.

Now let $H \in \mathcal{H}$ be chosen so that $\sum_{k=1}^{3} k_i(H)$ is as small as possible. We show that H is the required minor.

Let v_i, $i = 1, 2, 3$ be as described previously. Let S be the union of the node sets of the paths $P_i = \pi_H(yx_i)[y, v_i]$, $i = 1, 2, 3$, and the node set of each path in $\mathcal{B} = \{P :$ P is an H-external path whose attachment nodes lie on distinct P_i's$\}$. By (3.2), $|S| \geq 2$. Also, since G is cyclically 4-edge connected, it has an H-external path P_{uw} with $u \in S$ and $w \notin S$. Since $G[S]$ is connected, P_{uw} can be selected so that it has exactly one node, namely u, in common with the set $\cup_{i=1}^{3} V(P_i)$. We show that $w \notin V(\pi_H(\delta_H(y))$.

Suppose that $u \in V(\pi_H(yx_i)[y, v_i])$. If $w \in V(\pi_H(yx_j))$, for $j \neq i$, the definition of v_j is violated. Also, if $w \in V(\pi_H(yx_i))$, then $\mathcal{H}$ would contain a minor H' with $\sum_{i=1}^{3} k_i(H') < \sum_{i=1}^{3} k_i(H)$. It can be obtained from H by simply letting $\pi_{H'}(e) := \pi_H(e)$, $e \neq yx_i$, and $\pi_{H'}(yx_i) := \pi_H(yx_i)[y, u] \cup P_{uw} \cup \pi_H(yx_i)[w, x_i]$.

To conclude, suppose that (i) does not occur and so P_{uw} has attachment edges yx_i and some other edge incident to x_i. If P_{uw} does not intersect any H-external path of $\mathcal{B}$, then by choice of v_i, there is an H-external path with one end v_i and the other in the expansion of an edge in $\delta_H(y) \backslash \{yx_i\}$ and so (ii) occurs. So suppose that P_{uw} intersects a path Q in $\mathcal{B}$ with one attachment node w' in P_j for $j \neq i$. Let Q' be the subpath of Q from w' to the first node in common with P_{uw}. Since (i) does not occur, we must have that $Q' \cup P_{uw}$ satisfies (iii). $\square$

Corollary 3.3 *If G is a cyclically 4-edge connected cubic graph and $T \not\cong G, K_4$ a simple cubic minor of G, then G has a minor $H \cong T$ and a distant H-external path.*

by a triple (H, f, π) where (i) H is a graph (called the *graphical part*) (ii) f is a one-to-one mapping which makes correspond with each node v of H a node $f(v)$ of G, and (iii) π is a one-to-one mapping which makes correspond with each edge $e = uv$ of H a path $\pi(e)$ of G connecting $f(u)$ and $f(v)$. Moreover, for any two edges x_1x_2, y_1y_2, $V(\pi(x_1x_2)) \cap V(\pi(y_1y_2)) = \{f(x_1), f(x_2)\} \cap \{f(y_1), f(y_2)\}$.

Since we are only interested in minors up to isomorphism, we shall find it convenient to assume that the nodes of a minor H have the same labels as the corresponding nodes of G. In other words, for $v \in V(H)$, $f(v) = v$. Thus, if U is a subset of $V(G)$, and H is a cubic graph with $V(H) = U$, then H is a minor of G if for each edge uv of H there corresponds a (u, v)-path $\pi(uv)$ of G, called the *expansion* of uv, such that for any two edges x_1x_2, y_1y_2 of H, $V(\pi(x_1x_2)) \cap V(\pi(y_1y_2)) = \{x_1, x_2\} \cap \{y_1, y_2\}$. Thus we define our minors as consisting of an ordered pair (H, π). We will sometimes identify the minor with its graphical part H but it should be understood that there is an underlying map π_H associated with H.

Let H be a minor of a graph G. An edge e of H is said to be *short* if $\pi(e)$ has exactly one edge; otherwise e is *long*. For a subgraph F of H, or $F \subseteq E(H)$, we denote by $\pi(F)$ the graph $\cup_{e\in E(F)}\pi(e)$. A subgraph L of G is *H-external*, if it is connected and its edge set is contained in $E - E(\pi(H))$. The *attachment nodes* of L are the elements of $V(L) \cap V(\pi(H))$. The *attachment edges* of L, denoted by $H \triangleright L$, are the edges of H whose expansions contain an attachment node of L. When L is a path with ends v and w and $V(L) \cap V(\pi(H)) = \{v, w\}$, then L is also called an *H-external* path of G. An H-external path L is called *H-distant* if $H \triangleright L$ consists of two independent edges. A 3-*bridge* is an H-external subgraph B which is a tree with three leaves where the degree one nodes lie on distinct attachment edges and such that $|V(B) \cap V(\pi(H))| = 3$.

3.2 Distant Paths

Let G be a (2-connected) cubic graph with a cubic minor $T \ncong G$. Then T must clearly have long edges, and hence G must have T-external paths. In general this does not imply that there exist T-distant paths. When G is a cyclically 4-edge connected and $T \ncong K_4$, however, it is possible to alter T slightly to obtain a minor H isomorphic to T such that G has H-distant paths. This result plays a central role in establishing the structure of cyclically 4-edge connected cubic graphs without certain cubic minors. We shall derive it as a corollary of the following lemma. To present the lemma we need the following definitions.

We use the notation P_{xy} for a path P of a graph G to signify the fact that x and y are the ends of P. Also, for two nodes v and w of P, $P[v, w]$ will denote the segment of P from v to w. Let H be a minor of G. Two H-external paths of G, $P_{v_1v_2}$ and $P_{w_1w_2}$ *cross* on an edge xy of H if, x, w_1, v_1, y appear in this order on $\pi(xy)$, and $v_2 \in V(\pi(\delta_H(x)\backslash xy))$, $w_2 \in V(\pi(\delta_H(y)\backslash xy))$ (see Figure 4).

Lemma 3.1 *Let G be a cyclically 4-edge connected cubic graph and let $T \ncong G$ be a simple cubic minor of G. Then G has a minor $H \cong T$ which has one of the following:*

(i) *an H-distant path,*

(ii) *two H-external paths which cross on an edge of H, (see Figure 4), or*

(iii) *an H-external subgraph whose attachment edges form a triangle of H (see Figure 5).*

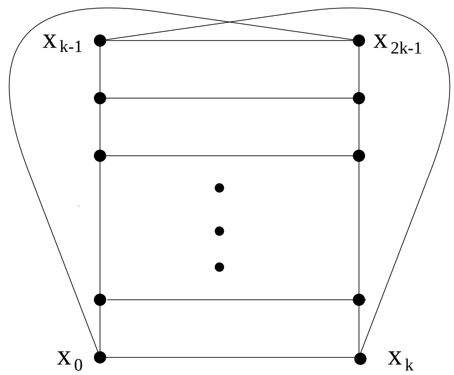


Figure 3: A Möbius k-ladder

not 3-edge-colourable. This leads us to propose that the same fractional-implies-integral property holds for the larger class of $P_{10}\backslash v$-free graphs.

Conjecture 2.2 ((1,2,3) Conjecture) *A $P_{10}\backslash v$-free graph is 3-edge-colourable if and only if it is fractionally 3-edge-colourable.*

Note that minor-minimality of $P_{10}\backslash v$ implies that this would be the largest possible (minor-closed) class for which 3-edge-colourability is determined by solving the fractional relaxation. We show that this conjecture is in fact implied by Grötzsch's Conjecture:

Theorem 5.2 *Any minimal counterexample to Conjecture 2.2 is planar.*

Of course forbidding $P_{10}\backslash v$ as a minor is quite restrictive in terms of 3-edge-colourability since many subdivisions of this graph are 3-edge-colourable. It would be too much to expect, however, a similar theorem with the word 'minor' replaced by 'subgraph' and $P_{10}\backslash v$ replaced by any finite list of graphs. If such were the case, then we could determine in polynomial time, whether a graph is 3-edge-colourable - an NP-complete problem, see [8] - by determining its fractional colourability and then checking for a finite number of subgraphs, which may be done in polynomial time. It may well be the case that such a list of *forbidden subgraphs* could be found for a restricted class of graphs. For example, it can be shown that the graphs used in Holyer's reduction will in general contain a Petersen minor. Can then the 3-edge-colourability of P_{10}-free graphs be determined in polynomial time? What is the list of forbidden subgraphs for P_{10}-free (or even prayer-free, see Section 4.1) graphs?

3 Minors

3.1 Notation

The fact that we are concerned only with minors of maximum degree three allows us to make their description simpler. Indeed, if H is a maximum degree three minor of a graph G, then G contains a subdivision of H as a subgraph. Thus, a *minor* of G is specified

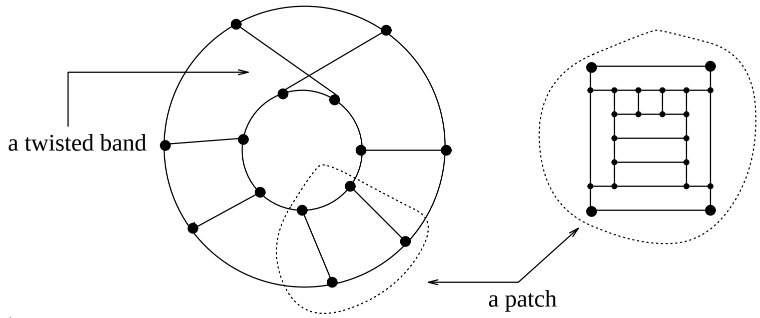


Figure 2: A patched ring

For $k \geq 5$, a *k-ring* is a graph obtained from a pair of disjoint k-circuits $C_1 = (x_0, \ldots, x_{k-1})$ and $C_2 = (y_0, \ldots, y_{k-1})$ by adding the edges $M = \{x_i y_i : i = 0, \ldots, k-1\}$. The edges of M are the *links* of the k-ring. To simplify notation, we shall assume that the labelling of the nodes of C_1 and C_2 is done counterclockwise. For $k \geq 2p > 0$, a *p-twisted k-ring*, denoted by $\Omega(p,k)$, is a graph obtained from a k-ring (and hence $k \geq 5$) by 'twisting' p pairs of neighbouring links. More precisely, it is obtained by identifying p disjoint subsets of M of the form $\{x_i y_i, x_{i+1} y_{i+1}\}$ and for each subset replacing its edges by $\{x_i y_{i+1}, y_i x_{i+1}\}$ (arithmetic modulo k). A *simple link* is an edge of the form $x_i y_i$; an edge of the form $x_i y_j$, $i \neq j$ is a *non-simple link*. A *twisted band* is a 4-circuit induced by the ends of two non-simple links. A 4-circuit induced by the ends of two simple links is a *simple band*.

A 1-*patch* is a (planar) graph which is obtained from a 4-circuit, with an independent pair (i.e., horizontal or vertical pair) of its edges initially labelled *active*, by subdividing the active edges an equal number of times and joining the resulting degree two nodes by non-crossing (or 'parallel') edges. A 2-*patch* is a planar graph obtained recursively from a 1-patch by changing the active circuit to one of the newly created 4-circuits and changing the "orientation" of the active edges (i.e., from horizontally active to vertically active edges (and vice versa)). A *patched ring* is a graph obtained from a twisted ring H by replacing one of its simple bands by a 2-patch and in the case when $p = 1$, also replacing its twisted band by a 1-patch (see Figure 2) with its links active.

A *Möbius k-ladder*, denoted by $M(k)$, is a circuit $C = (x_0, x_1, \ldots, x_{2k-1})$ with the *rung* edges $x_i x_{i+k}$ $i = 0, .., k-1$, see Figure 3.

We now state the main structural theorem of the paper.

Theorem 4.6 *Every cyclically 4-edge connected, cubic, $(P_{10} \backslash e)$-free graph G is either planar, a Möbius ladder or a patched ring.*

Theorem 4.6 along with the 4-node-colourability of planar graphs, is used to prove Tutte's Conjecture for cubic graphs with no minor isomorphic to the Petersen Graph with an edge removed (Section 5):

Theorem 5.1 *Every $P_{10} \backslash e$-free 3-graph is 3-edge-colourable.*

We also consider Grötzsch's Conjecture. It is routine to check that $P_{10} \backslash v$ is a minor-minimal graph with respect to the property of being fractionally 3-edge-colourable but perfect matchings. Later Lovász developed a theory which gave a complete characterization of this lattice [10].

Maximum Degree Three Graphs

An alternative generalization of the Four Colour Theorem is obtained in dropping the condition of regularity. It asserts (see [12]) the equivalence of fractional and integral 3-edge-colourability in planar graphs.

Conjecture 1.4 (Grötzsch) *A planar graph is 3-edge-colourable if and only if it is fractionally 3-edge-colourable.*

It follows easily from Theorem 1.1 that Grötzsch's Conjecture can be restated as:

Conjecture 1.5 *Subdividing $k \neq 1$ independent edges of a planar 3-graph results in a 3-edge-colourable graph.*

Note that if the conjecture is true, then the 3-edge-colourability of a planar graph could be determined simply by solving a linear program.

Finally, we mention that Grötzsch's conjecture is a special case of a conjecture of Seymour [11] which asserts that the matching polyhedron of any planar graph has the integer decomposition property. (A polytope P has the *integer decomposition property* if for any $x \in P$ and any integer k such that kx is an integral vector, x is expressible as the sum of k integer vectors in P. This implies that P is integral.) It is plausible that the matching polyhedron of each $P_{10}\backslash v$-free cubic graph has this property. Some evidence to support this is given in Theorem 5.2.

2 The Results

Our aim is to find decompositions along the lines of Wagner's classical result [15] for K_5-free graphs and the following well-known result:

Theorem 2.1 *Every 3-node connected $K_{3,3}$-free graph is planar except for K_5.*

Let $\mathcal{L}$ denote the class of P_{10}-free cubic graphs. In the present paper we restrict ourselves to cubic graphs by studying several subclasses of $\mathcal{L}$ which contain the class $\mathcal{P}$ of planar cubic graphs. To this end, let $P_{10}\backslash e$ denote the graph obtained by deleting an edge of the Petersen Graph (recall that P_{10} is edge-transitive) and let $P_{10}\backslash v$ denote the graph obtained by deleting a node of P_{10}. We denote by $\mathcal{L}^{-e}$ (respectively $\mathcal{L}^{-v}$, $\mathcal{K}^{+}$) the class of $P_{10}\backslash e$-free (respectively $P_{10}\backslash v$-free, $K_{3,3}$-free) cubic graphs. Thus

$$\mathcal{P} \subseteq \mathcal{K}^{+} \subseteq \mathcal{L}^{-v} \subseteq \mathcal{L}^{-e} \subseteq \mathcal{L}.$$

We refer the reader to results on related minors in triangle-free graphs whose statements can be found in [9].

The main purpose of this paper is to give decompositions for the classes $\mathcal{L}^{-v}$ and $\mathcal{L}^{-e}$. We defer the statement of the former until Section 4. The introduction of the latter result requires some definitions.

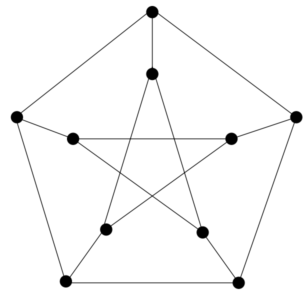


Figure 1: The Petersen Graph P_{10}.

of matchings in the graph and 1^T is the transpose of the vector of all ones. The work of Edmonds [4] yields the following combinatorial characterization:

Theorem 1.1 *Let G be a graph. Then $\eta^*(G) \leq 3$ if and only if G has maximum degree three and contains no induced subgraph which is obtained from a cubic graph by subdividing a single edge once.*

For a cubic graph G, this asserts that $\eta^*(G) \leq 3$ if and only if G has no cut edge; such graphs will be referred to as 3-*graphs* if they are also connected.

3-Graphs

We begin by stating Tait's reformulation [13] of the Four Colour Problem.
Four Colour Theorem *Every planar* 3*-graph is* 3*-edge-colourable.*
This was proved in [2], [3]. We are particularly interested in the following well-known stronger conjecture of Tutte [14].

Conjecture 1.2 (Tutte) *Every P_{10}-free* 3*-graph is* 3*-edge-colourable.*

We briefly mention that the existence of a 3-edge-colouring in cubic graphs immediately implies the existence of a cycle double cover and that the existence of such cycle covers has been shown true in its own right for P_{10}-free graphs ([1]). A predecessor to this result can be found in [5] and other interesting insights into related results may be found in [7].

Recall that a 3-edge-colouring of a graph is equivalent to expressing its edge incidence vector χ^E, as the sum of three incidence vectors of perfect matchings. In a classical paper of Seymour [11], a relaxation of this edge-colouring problem for P_{10}-free graphs is considered. Namely, Conjecture 1.2 is verified in case that the nonnegativity constraint is dropped.

Theorem 1.3 (Seymour) *If $G = (V, E)$ is cubic, P_{10}-free and has no cut edge, then there exist perfect matchings $M_1, \ldots, M_k$ and integers $\lambda_1, \ldots, \lambda_k$ such that*

$$\chi^E = \sum_{i=1}^{k} \lambda_i \chi^{M_i}.$$

Seymour's result is equivalent to showing that the incidence vector of the edge set of a bridgeless P_{10}-free graph is contained in the lattice generated by the incidence vectors of

Excluding Minors in Cubic Graphs

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Abstract. Let $P_{10}\backslash e$ be the graph obtained by deleting an edge from the Petersen graph. We give a decomposition theorem for cubic graphs with no minor isomorphic to $P_{10}\backslash e$. The decomposition is used to show that graphs in this class are 3-edge-colourable. We also consider an application to a conjecture due to Grötzsch which states that a planar graph is 3-edge-colourable if and only if it is fractionally 3-edge-colourable.

1985 *Mathematics Subject Classification:* 05C50,05C75.
Key Words and Phrases: Planar graph, Petersen Graph, Four Colour Problem, cubic, matching polyhedron, integer decomposition property, edge-colouring.

1 Background

We consider loopless graphs $G = (V, E)$ with node set V and edge set E. For an edge $e \in E$, we denote by G/e the graph obtained by *contracting* the edge e, i.e., identifying its *ends* and deleting the resulting loop. A *subgraph* (respectively *induced subgraph*) of G is a graph obtained by deleting nodes or edges (respectively deleting nodes) of G. Informally, a *minor* of G consists of (i) a graph H obtained by deleting nodes and contracting or deleting edges (called the *graphical part* of the minor) and (ii) a 'record' of the nodes or edges which were contracted or deleted (see Section 3.2 for a precise definition). We will often simply refer to H as being the minor when we are not concerned with how it was obtained from G. For a graph H, G is said to be H*-free* if no minor of G is isomorphic to H. For a proper subset S of the nodes of G, denote by $\delta(S)$ the set of edges with exactly one end in S. Such a set of edges is called an *edge cut* and if $|S| = 1$ or $|V| - 1$, then it is called *trivial*; otherwise it is *proper* or *nontrivial*. A graph is k*-edge connected* if $|\delta(S)| \geq k$ for each proper subset S of V and a cubic graph is *cyclically* 4*-edge connected* if each nontrivial edge cut has at least four edges. A cubic graph which is not cyclically 4-edge connected will be called *reducible*.

Let P_{10} denote the *Petersen Graph* (see Figure 1). This paper is concerned with the structure of cubic graphs which do not contain specified subgraphs of P_{10} as minors. The results are motivated in particular by certain conjectures about when a graph G possesses a 3*-edge-colouring*, that is, a partition of its edge set into three matchings. A trivial necessary condition for the existence of a 3-edge-colouring is that the *fractional edge-colouring number*, denoted by $\eta^*(G)$, is at most 3. $\eta^*(G)$ is defined as the value of the linear program $min\{1^T y : y^T S \geq 1^T,\ y \geq 0\}$ where the rows of S are the incidence vectors