Applying Lehman's Theorems to Packing Problems

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Abstract. A 0-1 matrix A is *ideal* if the polyhedron $Q(A) = conv\{x \in \mathbf{Q}^V : A \cdot x \geq 1, x \geq 0\}$ (V denotes the column index set of A), is integral. Similarly a matrix is *perfect* if $P(A) = conv\{x \in \mathbf{Q}^V : A \cdot x \leq 1, x \geq 0\}$ is integral. Little is known about the relationship between these two classes of matrices. We consider a transformation between the two classes which enables us to apply Lehman's modified theorem about deletion-minimal non-ideal matrices to obtain new results about packing polyhedra. This results in a polyhedral description for the stable set polytopes of *near-bipartite graphs* (the deletion of any neighbourhood produces a bipartite graph). Note that this class includes the complements of line graphs. To date, this is the only natural class, besides the perfect graphs, for which such a description is known for the graphs and their complements. Some remarks are also made on possible approaches to describing the stable set polyhedra of quasi-line graphs, and more generally claw-free graphs. These results also yield a new class of *t-perfect* graphs.

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1 Introduction

1.1 Ideal Matrices

A 0-1 matrix A is *ideal* (or has the *max-flow min-cut property*) if

$$Q(A) = \{x \in \mathbf{Q}^V : x \geq 0,\ A \cdot x \geq 1\} \tag{1}$$

is an integral polyhedron (where V denotes the index set of columns). It is clear that if A is ideal, then so is the matrix obtained by removing any dominating rows of A. Hence we restrict our attention to matrices without dominating rows. The *blocking matrix* of A, denoted by $b(A)$, is the matrix whose rows consist of all minimal 0-1 vectors in $Q(A)$. It follows ([8]) that $b(b(A)) = A$.

Evidently, $Q(A)$ is full dimensional. Furthermore, if A has no zero column, then the only nonnegative solution x to $A \cdot x = 0$ is the zero vector. Thus $Q(A)$ is pointed if A has no zero column.

For $c \in \mathbf{Q}_+^V$, we denote by $\nu_A^*(c)$ (or simply $\nu^*(c)$) the value $\min\{c \cdot x : x \in Q(b(A))\}$ and $\nu_A(c)$ the same minimum over the 0-1 vectors in $Q(b(A))$. We define $\tau_A^*(c) = \nu_{b(A)}^*(c)$ and similarly for $\tau_A(c)$. Thus A is ideal if and only if $\tau_A^*(c) = \tau_A(c)$ for each $c \in \mathbf{Q}_+^V$ - see [26].

For c all ones, we denote these parameters by ν_A, τ_A (or simply ν, τ) etc., i.e., the minimum number of ones in a row of A and $b(A)$ respectively. We also denote by r_A (or simply r) the integer $\max\{0, \nu\tau - |V|\}$.

For node $v \in V$, the *contraction* of v, denoted by A/v, is the matrix obtained by removing v's column and deleting any dominating rows. This corresponds to restricting to the face of $Q(A)$ obtained by setting $x_v = 0$ i.e., $Q(A/v) = Q(A) \cap \{x : x_v = 0\}$. The matrix obtained by the *deletion* of v, denoted by $A \backslash v$ is obtained by deleting v's column and keeping only those rows which had a 0 in v's position. This corresponds to restricting to the face of $Q(A)$ obtained by setting $x_v = 1$ i.e., $Q(A \backslash v) = Q(A) \cap \{x : x_v = 1\}$. It is straightforward to check that for $u, v \in V$, $A/v \backslash u = A \backslash u/v$. These two operations are also closed under forming blockers:

Proposition 1.1 (Seymour [27]) *For a matrix A and $v \in V$, $b(A/v) = b(A) \backslash v$ and $b(A \backslash v) = b(A)/v$.*

A *minor* of A is any matrix of the form $A \backslash S/T$ for disjoint $S, T \subseteq V$. A *c-minor* (respectively *d-minor*) is such a matrix with $S = \phi$ (respectively $T = \phi$).

Proposition 1.2 (Seymour [27]) *If A is ideal, then so too is any minor of A.*

We also have the following result which resembles the Perfect Graph Theorem (see [18]).

Theorem 1.3 (Lehman) *A matrix A is ideal if and only if $b(A)$ is ideal.*

1.2 Minimally Non-ideal Matrices

A *minimally non-ideal matrix* is a matrix which is not ideal such that each of its proper minors is ideal. Note that Theorem 1.3 implies that a matrix is minimally non-ideal if and only if its blocking matrix is minimally non-ideal. Since ideal matrices are closed under taking minors, there is a forbidden minor characterization of the class of all such matrices. This is equivalent to characterizing the minimally non-ideal matrices. There are three known infinite classes of minimally non-ideal matrices. The first two are C_n and $b(C_n)$ for each odd $n \geq 3$ (here C_n denotes the $n \times n$ cycle matrix). The other infinite class of minimally non-ideal matrices is the set of *degenerate projective planes*: for $k \geq 2$, F_k denotes the matrix with $V = \{0, 1 \ldots k\}$ and rows $\chi^{\{0,1\}}, \chi^{\{0,2\}}, \ldots \chi^{\{0,k\}}, \chi^{\{1,2,\ldots k\}}$. We call a matrix *nondegenerate* if it does not contain F_k as a minor for any $k \geq 3$. There are many other known minimally non-ideal matrices; the reader is referred to [6]. Lehman [16] and [17] shows that each nondegenerate minimally non-ideal matrix is regular in a way reminiscent of the "checkerboard" conditions given by Padberg [20] for minimally imperfect graphs. In the following, we denote by I (respectively J), the identity matrix (respectively matrix of all 1's) of appropriate dimension.

Theorem 1.4 (Lehman's Theorem) *Let A be a nondegenerate minimally non-ideal matrix. Then r_H satisfies $0 < r_H < \min\{\tau, \nu\}$ and there are exactly $|V|$ (linearly independent) rows of A with ν ones, $\{R_1, \ldots R_{|V|}\}$ and exactly $|V|$ (linearly independent) rows of $b(A)$ with τ ones $\mathcal{Z} = \{Z_1, \ldots Z_{|V|}\}$. Moreover, if we let R, Z be the square matrices whose rows are the vectors R_i and Z_i respectively, then the following hold:*

$$R \cdot Z = r_H I + J \tag{2}$$

Each column of R has ν ones; each column of Z has τ ones. (3)

Alternative proofs of this result are found in [28] and a more polyhedral approach is given in [22] and also by Fonlupt [9]. It is straightforward to see that Lehman's result also implies the existence of a unique fractional vertex for the associated polyhedron (cf [21]).

Corollary 1.5 *If A is minimally non-ideal, then $Q(A)$ has a unique fractional vertex. Furthermore, if A is nondegenerate, then this vertex is $\frac{1}{\nu}1$, and it has exactly $|V|$ neighbours, each of which corresponds to a vector of $b(A)$ with τ ones.*

1.3 Deletion-Minimal Matrices

We now state another result due to Lehman which is an amendment of the last theorem to matrices which are only deletion-minimal. A square 0-1 matrix A is said to be *Lehman* if for any $a_{ij} = 0$, the number of ones in column j is the same as the number of ones in row i. Note that any Lehman matrix may be written as:

$$\begin{bmatrix} A_1 & 1 & \dots & 1 \\ 1 & A_2 & & 1 \\ & \vdots & \dots & 1 \\ 1 & 1 & \dots & A_k \end{bmatrix}, \tag{4}$$

where each *block* A_i is a matrix with a constant number of ones in each row and column and k is maximal (i.e., no block can be further partitioned into two matrices). We denote by S_i the set of columns corresponding to the i^{th} block. A vertex x of $Q(A)$ has the *Lehman property* if there is a square nonsingular Lehman submatrix A_x of A such that $A_x \cdot x = 1$ and x is zero for any column not appearing in the matrix A_x. A_x is called a *defining matrix* for x. The next theorem is proved in [17].

Theorem 1.6 *(Lehman) If A is d-minimal, then each fractional vertex of $Q(A)$ has the Lehman property.*

The following result is implied by the fact that *every* vertex of $Q(A)$ for a d-minimal matrix is a Lehman vertex.

Corollary 1.7 *Let x^* be a fractional vertex of $Q(A)$ for a d-minimal matrix A and A_{x^*} be a defining Lehman matrix for x^*. Then for each block A_i of the matrix A_{x^*}, the matrix $A/(V - S_i)$ has precisely $|S_i|$ rows with a minimum number of ones and these are the rows of A_i. In particular, x^* has a unique defining matrix.*

2 Covering and Packing Polyhedra

A 0-1 matrix A is called *perfect* (see [20]) if $P(A) = \{x \geq 0 : A \cdot x \leq 1\}$ is integral. Perfect matrices have been extensively studied because of their ties to the Strong Perfect Graph Conjecture and theoretical integer programming in general. It would seem that there should also be a link with ideal matrices. It is true that many of the theorems for perfect matrices have analogues for the study of ideal matrices. On the other hand, the results for ideal matrices usually tend to have significantly more difficult proofs. It would be of interest to develop a theory for a common generalization of these two concepts. It would also be useful to find a natural bijection between the two classes of matrices.

In this section we relax the definition of perfection and idealism in order to find such a bijection between two larger classes. We use the bijection to also give a polyhedral description of the stable set polytopes of near-bipartite graphs in the following section.

For a 0-1 matrix A denote by $\tilde{A}$, the matrix $J - A$ and let $P^+(A)$ denote $P(A) \cap \{x : 1 \cdot x \geq 1\}$. The only vertices of $P(A)$ contained in $\{x : 1 \cdot x \leq 1\}$ are the standard vectors e_i and the zero vector. For if x^* is such a non-zero vertex, then there is a square nonsingular submatrix A_1 of A whose columns correspond to the support of x^* and for which $A_1 \cdot x^* = 1$. Since $1 \cdot x^* \leq 1$, each row of A_1 has all ones and so by nonsingularity A_1 must be a 1×1 matrix. Thus if we set $\mathcal{B} = \{x \in P(A) : 1 \cdot x = 1\}$ and call this a trivial facet of $P(A)$ we have:

- The only vertices of $P(A)$ in $\{x : 1 \cdot x \leq 1\}$ are 0 and $e_1, \ldots, e_n$
- The nontrivial facets of $P(A)$ and $P^+(A)$ are precisely the same
- There is a $1-1$ correspondence between nonzero vertices of $P(A)$ and vertices of $P^+(A)$.

We define the function $\Phi : \mathbf{R}^V \backslash \mathcal{B} \rightarrow \mathbf{R}^V$ by $\Phi(x) = \frac{x}{1 \cdot x - 1}$. We remark that this function is analagous to the transformation given by Laurent ([13]) except that in this context, the function is applied directly to the polyhedron (i.e., to the polyhedron instead of its antiblocker). Note that $\Phi^{-1} = \Phi$ and that for any vectors $x_1, \ldots, x_k$, and scalars $c_1, \ldots, c_k$: if $\sum_i c_i x_i = 0$, then $\sum_i (c_i(1 \cdot x_i - 1))\Phi(x_i) = 0$. Hence

$$\Phi \text{ preserves the linear independence of a set of vectors.} \tag{5}$$

Theorem 2.1 *If A has no column of zeros, then Φ is a bijection between $P^+(A) - \mathcal{B}$ and $Q(\tilde{A})$ which preserves tight inequalities (of corresponding rows of A and $\tilde{A}$). And if $F \not\subseteq \mathcal{B}$, then F is a face of $P^+(A)$ if and only if $\Phi(F)$ is a face of $Q(\tilde{A})$. Furthermore $dim(F) = dim(\Phi(F))$. In particular, there is a $1-1$ correspondence between the nontrivial vertices of $P^+(A)$ and the vertices of $Q(\tilde{A})$.*

Proof: Note that A having no column of zeros implies that $1 \cdot x > 1$ for each $x \in Q(\tilde{A})$. Direct calculation shows that $x \in P^+(A) - \mathcal{B}$ if and only if $\Phi(x) \in Q(\tilde{A})$. In fact for any 0-1 vector a (which is not all 1's) and any $x \geq 0$, $a \cdot x = 1$ if and only if $\tilde{a} \cdot \Phi(x) = 1$. Hence if $F \not\subseteq \mathcal{B}$, is a face of $P^+(A)$, then there is a row partition of A into submatrices A_1, A_2 such that F is the set of vectors satisfying $1 \cdot x \geq 1$, $A_1 \cdot x \leq 1$, $A_2 \cdot x = 1$. Thus $F' = \{\Phi(x) : x \in F - \mathcal{B}\}$ is the set of vectors in $Q(\tilde{A})$ such that $\tilde{A}_2 \cdot x = 1$. Moreover, by (5), if $S \subseteq F'$ is a set of affinely independent vectors, then so is $\Phi(S)$, and so $dim(F) \geq dim(F')$. The reverse inequality also holds since a maximum affinely independent set of vectors in F can be found in $F - \mathcal{B}$. □

Corollary 2.2 *For $n > 1$, an $n \times n$ 0-1 matrix A is nonsingular if and only if $\tilde{A}$ is nonsingular.*

Call a matrix A *outer-perfect* (respectively *outer-ideal*) if every vertex of $P(A)$ (respectively $Q(A)$) is either 0-1 valued or of the form $\frac{1}{k-1}\chi^S$ for some $|S| = k$. We define *outer-ideal* matrices similarly. Then Theorem 2.1 implies:

Theorem 2.3 *If A is a 0-1 matrix with no column of zeros, then A is outer-perfect if and only if $\tilde{A}$ is outer-ideal.*

Proof: Suppose $\tilde{A}$ is outer-ideal. Then by Theorem 2.1, any vertex of $P(A)$ is either $0, e_1, \ldots, e_n$ or of the form $\Phi(x)$ where x is a vertex of $Q(\tilde{A})$. It follows that A is outer-perfect. The other direction of the proof is similar. □

Finally we mention an application to stable set polyhedra. For a graph G, its *stable set polytope*, denoted by $P(G)$, is the convex hull of incidence vectors of stable sets in G. Similarly the *node cover polyhedron*, is the dominant of the convex hull of incidence vectors of G's node covers (set complements of stable sets).

Corollary 2.4 *For a connected graph G, $a \cdot x \leq 1$ induces a nontrivial facet of $P(G)$ if and only if $a \cdot x \geq (1 \cdot a) - 1$ induces a nontrivial facet of the node cover polytope.*

Proof: Let A have rows corresponding to incidence vectors of maximal stable sets of G. Thus the rows of $\tilde{A}$ correspond to minimal node covers. Results of Fulkerson [10] imply that the nontrivial facets of $P(G)$ (respectively the node cover polyhedron) correspond to the maximal (respectively minimal) vertices of $P(A)$ (respectively $Q(\tilde{A})$). The result now follows from Theorem 2.1. □

One immediate consequence is the following.

Corollary 2.5 *A connected graph G is perfect if and only if its node cover polytope is $\{x : x \geq 0,\ x(K) \geq |K| - 1$ for each clique $K\}$.*

This also implies the simple result that the edge inequalities define $P(G)$ if and only if they also define (with $\geq$ replacing $\leq$) the node cover polyhedron if and only if G is bipartite.

3 Near-Bipartite Graphs

A graph $G = (V_G, E_G)$ is called *near-bipartite* if $G - N(v)$ is bipartite for each node v. We begin by considering the near-bipartite graphs which induce rank facets for their stable set polytope. We will see that each such graph has the following partitionable property. A subset $S \subseteq V$ is *patterned* if it contains exactly $|S|$ maximum stable sets of $G[S]$ whose incidence vectors are linearly independent, and each node of S is in α_G [1] of these sets. Moreover, S must be *non-separable*, that is, it cannot be partitioned into nonempty subsets S_1, S_2 so that each node of S_1 is adjacent to each node of S_2.

A web W_k^n is a graph with nodes $0, 1, \ldots, n-1$ whose edges are all of the form ij such that $|i - j| \leq k$ $(mod\ n)$. For example, W_1^n is a cycle of length n. Giles and Trotter [12] showed that there may be somewhat complex facet-inducing inequalities for the stable set polytope of such graphs. Their aim was to give evidence of the difficulty in describing the

[1] We use α_G, or simply α, to denote the size of a maximum stable set in G

stable set polytope of *claw-free* graphs, i.e., no node is adjacent to 3 independent nodes. In fact each web has a stronger property: the neighbourhood of each node partitions into two cliques. Graphs with this stronger property are called *quasi line graphs* and were the subject of investigation of the thesis by Ben Rebea [25]. Since the near-bipartite graphs are precisely the complements of quasi-line graphs, we have that each $\overline{W}_k^n$ is near-bipartite. We show that besides the clique inequalities, the rank facets of near-bipartite graphs correspond (essentially) to graphs $\overline{W}_{\alpha-1}^{2\alpha+1+k}$ for integers $\alpha \geq 2,\ k \geq 0$. Note that for $k = 0$, this results in an odd hole. For $k = 1$, this yields a graph where each node i is adjacent to the nodes $i+\alpha, i+\alpha+1, i+\alpha+2$ $(mod\ (2\alpha+2))$. If α is odd, this graph is an even Möbius $(\alpha+1)$-ladder (see Figure 1). Recall also from above that $\overline{W}_1^{2t+1}$ is an odd antihole (complement of an odd hole) for each $t \geq 0$.

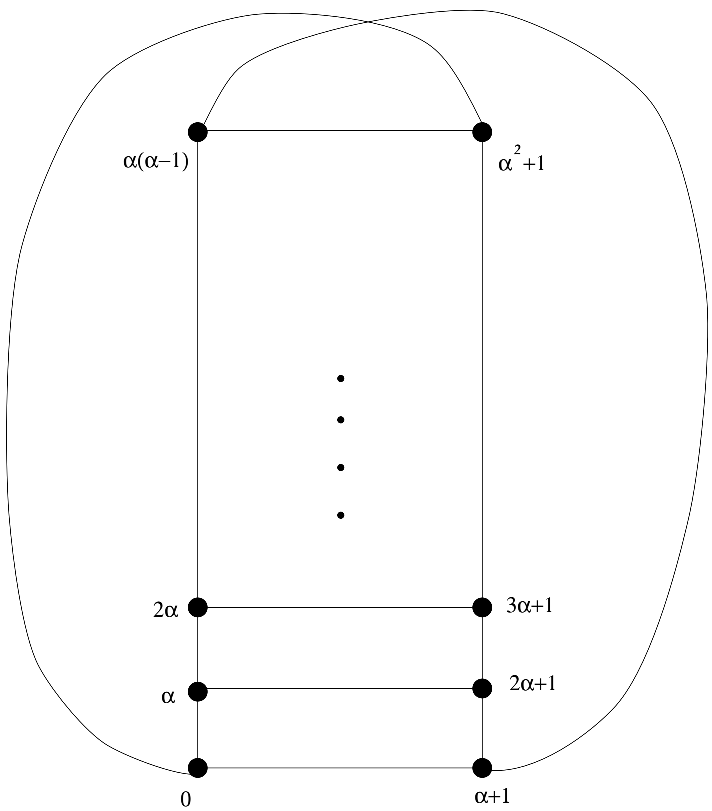

Figure 1: *Labelling (modulo $2\alpha + 1 + k$) of nodes for an even Möbius Ladder.*

We begin by showing that each patterned set induces a regular graph.

Proposition 3.1 *If G is near-bipartite and for each v, $G - N[v]$ has a set of α affinely independent stable $\alpha - 1$-sets, then G is a $(|V| - 2\alpha + 1)$-regular graph.*

Proof: Let v be an arbitrary node and set $B = G - N[v]$. Then $\alpha_B = \alpha - 1$ and the affine hull F of the incidence vectors of B's maximum stable sets has dimension at least $\alpha - 1$. Let

$e_1, \ldots, e_k, v_1, ..., v_l$ be an $\alpha-1$ clique cover of B, where each e_i is an edge $u_i w_i$ and the v_i's are singletons. Since the inequalities $x_{v_i} \leq 1,\ x_{u_i}+x_{w_i} \leq 1$ form a linearly independent system of tight constraints for F, we have that $dim(F) \leq |V_B|-(k+l) = 2k+l-k-l = k \leq k+l = \alpha-1$. Thus $l = 0$ and so in particular B has a perfect matching. Thus we have proved that for each node of G, it has precisely $2(\alpha-1)$ non-neighbours and the result follows. □

Corollary 3.2 *If G is near-bipartite and S is a patterned subset, then $G[S]$ is $(|S|-2\alpha_{G[S]}+1)$-regular.*

Proof: Clearly we may assume that $S = V$. The result then follows from the fact that each node of G is in precisely α maximum stable sets and hence $G - N[v]$ satisfies the hypotheses of Proposition 3.1. □

For the following result, I am indebted to Andras Frank who enlightened me to the operation on two maximum stable sets which produces two new maximum stable sets; this was critical in the proof of the result.

Lemma 3.3 (Frank, Shepherd) *If G is bipartite and has precisely $\alpha+1$ maximum stable sets, and their incidence vectors are affinely independent, then G has a unique Hamilton path $1, 1', 2, 2', \ldots, \alpha, \alpha'$. Moreover, any other adjacency in the graph is of the form $i'j$ where $i < j$.*

Proof: As in the proof of Proposition 3.1, we have that G contains a perfect matching M as well as a bipartition $V_1 \cup V_2$ where each V_i is a maximum stable set. Now for any two maximum stable sets S, T we define two new stable sets: $S \bigwedge T = (S \cap T \cap V_1) \cup (S \cap V_2) \cup (T \cap V_2)$, $S \bigvee T = (S \cap T \cap V_2) \cup (S \cap V_1) \cup (T \cap V_1)$. Note that these new stable sets are distinct from S, T unless $(S \cap V_1) \subseteq (T \cap V_1)$ or vice versa. Thus if this is not the case, then we have:

$$\chi^S + \chi^T = \chi^{S \bigvee T} + \chi^{S \bigwedge T}$$

contradicting the affine independence assumption. Thus for any pair of sets in $\mathcal{S} = \{S \cap V_1 : S \text{ is a maximum stable set}\}$, one of them contains the other. Since $|\mathcal{S}| = \alpha + 1$, it follows that there is an ordering of the nodes in V_1, $1, 2, \ldots, \alpha$ such that the sets in $\mathcal{S}$ are of the form $\{i : i < k\}$ for some $k = 1, \ldots \alpha+1$. Then if we let i' be the node matched to i in M we have that each maximum stable set of G is of the form: $X_k = \{i : i < k\} \cup \{j' : j \geq k\}$ for $k = 1, \ldots, \alpha+1$. We show that $P = 11'22'33' \ldots \alpha\alpha'$ is a Hamilton path in G. If this is not the case, then there exists some $i < \alpha$ such that $i'(i+1) \notin E$. But since each X_k is a stable set there is no edge mn' where $m < n$. Thus $i'(i+1) \notin E$ implies that $\{1, \ldots, i-1, i', i+1, (i+2)', \ldots, \alpha'\}$ is a maximum stable set whose intersection with V_1 is not of the prescribed form, a contradiction.

To see that P is unique, note first that the node 1 has degree one and so any Hamilton path must use the edge $11'$. Next note that $G - \{1, 1'\}$ again has α stable sets of size $\alpha - 1$ which are affinely independent. Thus by induction $22'33' \ldots \alpha\alpha'$ is its unique Hamilton path. The result now follows from the fact that $1'\alpha' \notin E$. □

The above result varies from most sufficient conditions for the existence of Hamilton paths in that it does not rely on the density of edges in the graph. For example, a simple path satisfies the conditions of the previous result.

We now prove the main structural lemma. We call a web $W_{\alpha-1}^{2\alpha+1+k}$ *prime* if $k \geq 0$ and $2\alpha+1+k$ and α are relatively prime.

Lemma 3.4 *If G is near-bipartite and V is a patterned set, then G is either a singleton or a prime antiweb $G = \overline{W}_{\alpha-1}^{2\alpha+1+k}$. Moreover, any prime antiweb induces a rank facet for its stable set polytope.*

Proof: If $\alpha = 1$, then clearly G is a singleton so assume that $\alpha \geq 2$. By Corollary 3.2, each node of G has exactly $2(\alpha-1)$ nodes to which it is not adjacent. Consider an arbitrary node and give it the label 0. By Lemma 3.3 there is a unique Hamilton path $P = x_1, y_1, x_2, y_2 \ldots x_{\alpha-1}, y_{\alpha-1}$ in $G - N[0]$. Also by this result, $x_1, y_{\alpha-1}$ each has exactly one neighbour in $V(P)$ and so these nodes are adjacent to $|N[0]| - 1$ nodes in $N[0]$ by regularity. We call these the *near-twins* of 0 and remark that each node v then has exactly two near-twins and moreover, these are the endpoints of the unique Hamilton path in $G - N[v]$. First label x_1 as 1 and then for each $i \geq 1$, give a node label $i+1$ if it is the unique near-twin of i which is not $i-1$. Evidentally this gives rise to an unambiguous labelling of (some of the) nodes $0, 1, \ldots, t$ such that for each i, the near-twins of i are $i-1, i+1 \pmod{t+1}$. (Note also that $y_{\alpha-1}$ must be the node labelled t.) We now claim that for each $i = 2, \ldots, \alpha-1$, the node x_i is labelled i. For let $G^1 = G - N[x_1]$, then 0 has degree precisely one in G^1 and also $y_{\alpha-1}$ has degree precisely one in $G - N[0]$. It follows that the only possible Hamilton path in G^1 is $x_2, y_2, \ldots, y_{\alpha-1}, v, 0$ where v is a neighbour of 0. From our preceding comments, x_2 receives the label 2. It now follows by induction that each x_i receives label i. Since this argument can be applied to any node, we see that each node i is non-adjacent to nodes $i+1, i+2, \ldots, i+\alpha-1$. Thus for each i, i is also non-adjacent to $i-1, i-2, \ldots, i-(\alpha-1)$. But i is non-adjacent to precisely $2(\alpha-1)$ nodes and so this accounts for all non-adjacencies. In particular, each node v appears in the list $0, \ldots, t$, otherwise it is adjacent to every node in $\{0, 1, \ldots, t\}$ and this contradicts non-separability of a patterned set. The labelling thus shows that G is in fact the complement of a web $W_{\alpha-1}^n$, where $n \geq 2(\alpha-1)+3$, as desired.

Now consider the graph H obtained from $V(W_{\alpha-1}^n)$ by adding the edges $i(i+\alpha)$ (modulo n) for $i = 0, \ldots, n-1$. Note that H is 2-regular (possibly with multiple edges) and consists of a single cycle if and only if n and α are relatively prime. For any cycle $C = (\pi_1, \pi_2, \ldots, \pi_l)$ of H, we have that $\sum \chi^{S_i}$ is a vector of 2's, where each S_i is the maximum clique $\{\pi_i, \pi_i + 1, \ldots, \pi_i + \alpha - 1\}$ of $W_{\alpha-1}^n$. Thus n, α must be relatively prime by the fact that $\overline{W}_{\alpha-1}^n$ has precisely n maximum stable set incidence vectors, and these vectors are linearly independent.

Finally, note that if $W_{\alpha-1}^n$ is a prime web, then the edges of H (as defined above) are α-critical in $\overline{W}_{\alpha-1}^n$ and form a spanning connected subgraph. Thus $\overline{W}_{\alpha-1}^n$ induces a rank facet (see [4]). □

An important class of graphs in the study of perfect graphs are the so-called *partitionable* graphs. G is partitionable if it has $\alpha\omega+1$ nodes and for each node v, $G-v$ has an ω-colouring and an α clique cover. Results in [2] show that each partitionable graph is patterned and hence we have the following result directly from Lemma 3.4.

Corollary 3.5 *Any partitionable quasi-line (near-bipartite) graph is a web (antiweb).*

Chvátal [3] has shown that the class of webs satisfies the strong perfect graph conjecture and hence the above corollary gives a new proof that the claw-free graphs satisfy the conjecture also (see [23]).

Corollary 3.6 *The only minimally imperfect claw-free graphs are the odd holes and odd antiholes.*

Proof: Suppose that G is claw-free and minimally imperfect. Then for each node, $G[N[v]]$ is perfect and also has no stable set of size 3. Thus $N[v]$ partitions into two cliques. Thus G is a quasi-line graph and the result follows from Corollary 3.5 and Chvátal's result. □

For pairwise disjoint node sets $\{G_1, \ldots G_k\}$ such that all possible edges exist between distinct C_i's, the *join* of the rank facets for these graphs is the inequality $\sum_{i=1}^{k} \frac{1}{\alpha_{G_i}} \chi^{V_{G_i}} \leq 1$. Note that this inequality is valid for $P(G)$.

Theorem 3.7 *If G is near-bipartite, then any nontrivial facet-inducing inequality is the join of clique and prime antiweb inequalities.*

Proof: Let A be a matrix whose rows are the incidence vectors of maximal stable sets in G and hence $\tilde{A}$'s rows correspond to the minimal node covers. From this it follows easily that $b(\tilde{A})$ is the edge-node incidence matrix of G. Thus for any node v, $b(\tilde{A})\backslash v$ is the edge-node incidence matrix of $G - N[v]$ and hence is ideal. Thus $\tilde{A}$ is deletion-minimal and so we may apply Theorem 1.6 and then Theorem 2.1 to $Q(\tilde{A})$. Now the facets of $P(G)$ correspond to the maximal vertices of $P(A)$ and by Theorem 2.1, each such vertex is of the form $\Phi(x^*)$ where x^* is a minimal vertex of $Q(\tilde{A})$. If x^* is integral, then $1 \cdot x^* = 2$ and so $\Phi(x^*)$ is also integral. Thus the only fractional vertices of $P^+(A)$ (and hence also of P(A)) are of the form $\Phi(x^*)$ where x^* is a fractional vertex of $Q(\tilde{A})$. Corollary 1.7 and Theorem 1.6 imply that any such x^* is the unique solution to a system $B \cdot x = 1$, where B is of the form (4) and each row of B corresponds to a minimum node cover of G. Thus $\Phi(x^*)$ is the unique solution to the system $\tilde{B} \cdot x = 1$ where each block, on column set C_i say, consists of incidence vectors of maximum stable sets in $G[C_i]$. Nonsingularity now implies that $\Phi(x^*) = \sum_i \frac{1}{\alpha_{G[C_i]}} \chi^{C_i}$. The result now follows from Lemma 3.4 and the fact that each C_i is patterned since Corollary 1.7 implies that $G[C_i]$ has exactly $|C_i|$ minimum node covers. □

The operation of joining gives a way of decomposing certain non-rank facet inequalities of $P(G)$ for a general graph into basic *facet blocks*. For instance, in the case of near-bipartite graphs, these blocks are cliques (or simply singletons) and the antiwebs. It would be of interest to understand better for which graphs, do all facets arise from such joining operations. Another example of such a class was given by Cook [5]. He showed that for any graph with no stable set of size three, its stable set polytope facets correspond to inequalities which are the join of a clique with a collection of graphs each of whose complement is non-bipartite - see [29].

Another consequence of these results is that for the antiwebs (other than odd holes and antiholes) which induce rank facet inequalities, the region defined by inequalities $x(S) \leq 1$ for each of their stable sets has a fractional vertex $\frac{1}{\alpha}$ which has precisely n neighbours each of which corresponds to a fractional vertex derived from some smaller antiweb. This seems to mirror Padberg's results at a different level since for minimally imperfect graphs, there is a unique fractional vertex with exactly n neighbours each corresponding to a maximum clique.

Line Graphs

One prominent class of near-bipartite graphs are the complements of line graphs. Recall that the *line graph* of a (multi-)graph G, denoted by $L(G)$, is a graph whose nodes are the edges of G and two vertices of $L(G)$ are adjacent if the corresponding edges in G are incident. Evidentally, each neighbourhood in $L(G)$ partitions into two cliques and hence $\overline{L(G)}$ is near-bipartite. It turns out that the polyhedral description for this class is considerably simpler. This is based on the following fact:

Fact 3.8 *A web is a line graph if and only if it is a cycle, clique or W_2^6.*

Proof: Clearly each cycle and clique is a line graph. Also W_2^6 is the line graph of K_4 so consider $G = W_k^n$ where $\alpha > 1$ and so $n \geq 2k+2$. Let F be the graph induced by $0, k-1, k, k+1, 2k, 2k+1$. If $k \geq 2$ and $n > 2k+2$, then F is one of the graphs in Figure 2 (a),(b), or (c). Each graph in this figure cannot exist as an induced subgraph in a line graph of some multigraph (cf Lovász,Plummer [19]). Thus $n = 2k+2$ and so each node i is adjacent to every other node except $i+k+1$. Thus F is obtained from Figure 2 (c) by adding the edge $(k-1)(2k+1)$. Now if $n > 6$, then any node outside of F is adjacent to each node of F and so G contains Figure 2 (d). Thus $G = W_2^6$. □

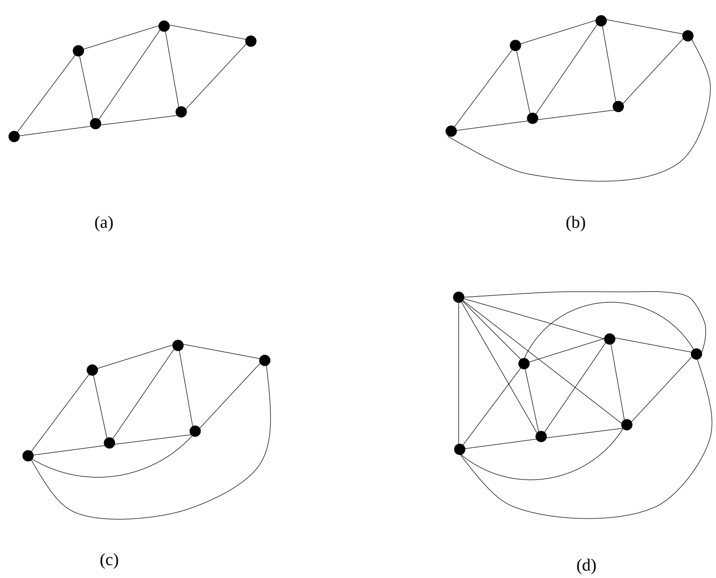

Figure 2: *Forbidden Line Graph Subgraphs.*

This fact together with Theorem 3.7 imply the following.

Theorem 3.9 *If G is the complement of a line graph, then each nontrivial facet of $P(G)$ is the join of a clique and some odd antihole inequalities.*

This can be restated in terms of the root graph G; note that the stable sets of $\overline{L(G)}$ correspond to triangles and subsets of 'stars' in G.

Theorem 3.10 *If G is a multigraph, then $P(\overline{L(G)})$ is*

$$\left\{ x \in \mathbf{Q}^{E(G)} : \begin{array}{l} x \geq 0 \\ x(M) + \frac{1}{2}\sum_{i=1}^{k} x(E(C_i)) \leq 1 \\ \qquad \textit{for each collection of node disjoint odd cycles} \\ \qquad C_i \textit{ (other than triangles) and disjoint matching } M \textit{ in } G \end{array} \right\}.$$

Theorem 3.10 together with results for antiblocking polyhedra gives also a linear description for the convex hull of triangle-free 2-matchings of a graph. The constraints in this system correspond to the stable sets in $P\overline{L(G)}$, namely the degree constraints $x(\delta(v)) \leq 1$ and a constraint $x(T) \leq 1$ for each *expanded* triangle, that is a set of edges whose underlying simple graph is a triangle. A *perfect* triangle-free 2-matching is one for which each node is incident to some edge. The convex of these perfect 2-matchings was given by Cornueéjols and Pulleyblank [7]. Note that the system is obtained from the linear description just described by setting each degree constraint to equality.

Other than the perfect graphs, this is the only natural class of graphs for which we have a polyhedral description for the stable set polytopes for the graphs in the class as well as their complements. It is well-known that the facets for $P(G)$ correspond to the maximal vertices of its antiblocker $A(P(G)) = \{z \geq 0 : z \cdot x \leq 1, \ \forall x \in P(G)\}$. In the case of perfect graphs this implies that $A(P(G)) = P(\bar{G})$ and $A(P(\bar{G})) = P(G)$. We ask if there is a relationship between $A(P(G))$ and $A(P(\bar{G}))$ which could be harnessed for other classes of graphs. This would be of potential use when we know a polyhedral description for each $P(G)$ but not for each $P(\bar{G})$. In particular such a relationship could be applied in studying quasi-line graphs given that Theorem 3.7 describes the stable set polytope facets for near-bipartite graphs - see Figure 3. The question of polyhedral descriptions for quasi-line graphs and more generally for claw-free graphs remains one of the interesting open problems in polyhedral combinatorics (see [25], [24], [12]).

We also remark that the inequalities for $P(\overline{L(G)})$ correspond to 2-matchings in G and can be obtained by rounding from the 'clique' constraints, i.e., constraints of the form $x(M) \leq 1$, for a matching M. Similarly the degree constraint polytope also has Chvátal rank one (yielding the matching polytope, described by Edmonds).

T-Perfect Graphs

A graph G is *t-perfect* if its stable set polytope denoted by $P(G)$, is defined by the edge inequalities ($x_u + x_v \leq 1$; $uv \in E_G$) and the odd cycle inequalities ($x(V(C)) \leq \alpha_{|C|}$; for each odd cycle C of G). The following is immediate from Theorem 3.7.

Corollary 3.11 *A near-bipartite graph G is t-perfect if and only if it has no odd wheels and no prime antiwebs other than odd holes.*

Gerards and Schrijver [11] showed that a graph with no subgraph which is an odd K_4 is t-perfect. An *odd* K_4 is any graph obtained by subdividing the edges of a K_4 in such a way, that the 'images' of the four triangles become odd cycles. Note that the class of t-perfect graphs given by Corollary 3.11 is not contained in the class of graphs with no odd K_4. For instance, subdivide once, each edge in a matching of K_4. A more elaborate example

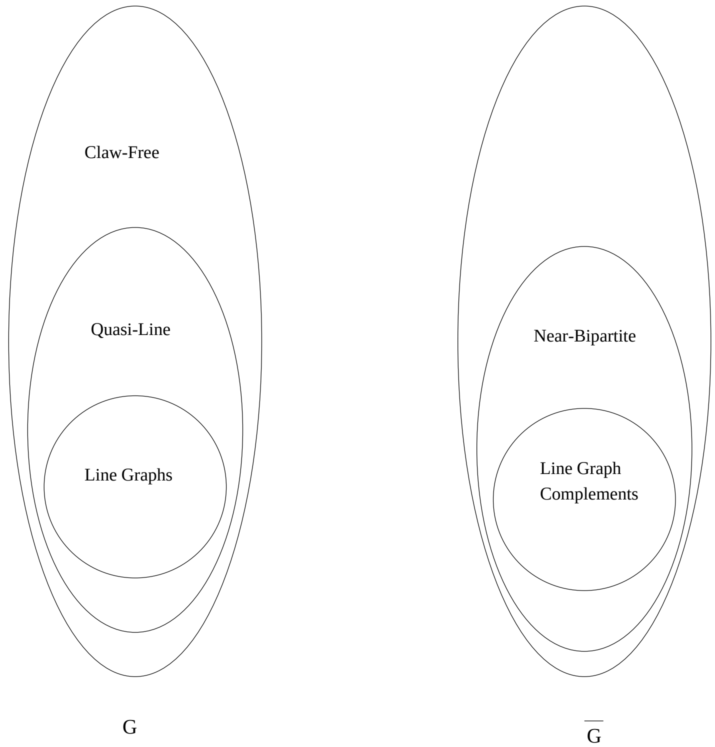

Figure 3:

(furnished by B. Gerards) of a t-perfect near-bipartite graph which contains an odd K_4 is depicted in Figure 4.

We can also deduce the following.

Corollary 3.12 • *The Petersen Graph is t-perfect*

- *Any even Möbius Ladder is minimally non-t-perfect.*

A well-known open problem concerning t-perfect graphs is whether the system of inequalities (edge, odd cycle and nonnegativity constraints) are *total dual integral*, that is, whether the corresponding linear program has a dual optimal solution for each integral objective function for the primal. Such graphs are called *dual t-perfect.*

Another question is to determine for which t-perfect graphs G, does $P(G)$ have the *integer decomposition property*: for each natural number k, any integral vector in $kP(G)$ is expressible as the sum of k vertices of $P(G)$ (see [1]). It is an easy consequence that any such graph G must be 3-node-colourable. On the other hand, Laurent, Schrijver and Seymour [14] have shown that the complement of the line graph of a prism (i.e., complement of a 6-cycle), is t-perfect but not 3-node-colourable. We note that the graph mentioned above is easily seen to be t-perfect by Theorem 3.10. In fact this theorem characterises all t-perfect complements of line graphs. This forms a fairly simple class since if $\overline{L(G)}$ is t-perfect, then G may not even have a matching of size 4. The class of t-perfect line graphs is more complex.

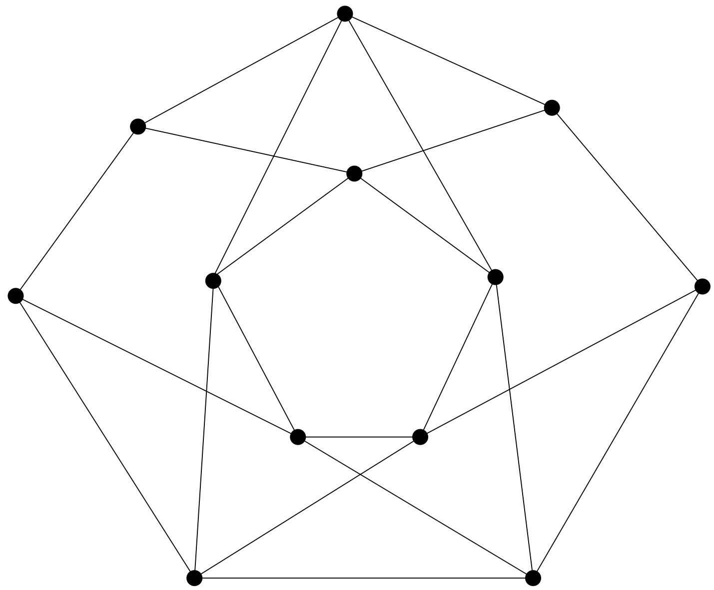

Figure 4: *Near-bipartite t-Perfect graph with an odd K_4.*

For each $k \geq 4$, however, it is not known whether each t-perfect graph is k-node-colourable. It is also not known if the stable set polytope of each 3-node-colourable t-perfect graph has the integer decomposition property.

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