Near-Optimal Hardness Results and Approximation Algorithms for Edge-Disjoint Paths and Related Problems[*]

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Abstract

We study the approximability of edge-disjoint paths and related problems. In the edge-disjoint paths problem (EDP), we are given a network G with source-sink pairs (s_i, t_i), $1 \le i \le k$, and the goal is to find a largest subset of source-sink pairs that can be simultaneously connected in an edge-disjoint manner. We show that in *directed* networks, for any $\epsilon > 0$, EDP is NP-hard to approximate within $m^{1/2-\epsilon}$. We also design simple approximation algorithms that achieve essentially matching approximation guarantees for some generalizations of EDP. Another related class of routing problems that we study concerns EDP with the additional constraint that the routing paths be of bounded length. We show that, for any $\epsilon > 0$, bounded length EDP is hard to approximate within $m^{1/2-\epsilon}$ even in undirected networks, and give an $O(\sqrt{m})$-approximation algorithm for it. For directed networks, we show that even the single source-sink pair case (i.e. find the maximum number of paths of bounded length between a given source-sink pair) is hard to approximate within $m^{1/2-\epsilon}$, for any $\epsilon > 0$.

1 Introduction

In the *edge-disjoint paths* problem, denoted EDP, we are given a (possibly directed) graph G and a set $\mathcal{T} = \{(s_i, t_i) : 1 \leq i \leq k\}$ of k source-sink pairs, and the objective is to connect a maximum number of these pairs via edge-disjoint paths. EDP turns out to be a fundamental, extensively studied problem in the fields of combinatorial optimization, algorithmic graph theory and operations research, and is one of the classical NP-hard problems [9]. This paper investigates the approximability of EDP and of two related classes of network routing problems (on undirected as well as directed graphs) that are natural generalizations and variants of EDP. We now define these classes of problems for the directed graph case; we omit defining their standard restrictions to the undirected case.

Multicommodity Flow Problems: In multicommodity problems, we are given a directed graph $G = (V, A)$ and a set $\mathcal{T}$ of k source-sink pairs as above. We let $m = |A|$ and $n = |V|$ throughout. In addition, we are given an integer capacity function $u : A \to \mathbf{Z}$ on the arcs, and a positive integer demand d_i for each (s_i, t_i) pair in $\mathcal{T}$, $1 \leq i \leq k$; d_i represents the bandwidth requested for flow from source s_i to sink t_i. The source-sink pairs need not be at distinct vertices. We use the notation u_{min} and d_{max} to denote the minimum capacity and maximum demand value respectively. In all versions discussed, we assume that the bandwidth assigned (or reserved) for a pair (s_i, t_i) induces a standard network flow between s_i and t_i of value d_i. Depending on the version addressed, we may require extra conditions on each of these s_i-t_i flows, e.g., for *unsplittable flow problems*, we require each s_i-t_i flow to be sent on a single path.

We denote by $\mathcal{P}_i$ the set of all (simple) directed paths in G from s_i to t_i. A *routing* (of $\mathcal{T}$) in G is an assignment $x : \mathcal{P} \to \mathbf{R}_+$ of weights to directed paths in G, where $\mathcal{P} = \cup_{1 \leq i \leq k} \mathcal{P}_i$. A routing is said to *fulfill* the demand d_i for pair i, if $\sum(x(P) : P \in \mathcal{P}_i) = d_i$, i.e, if the following demand constraint is satisfied with equality:

$$\sum(x(P) : P \in \mathcal{P}_i) \leq d_i. \tag{1}$$

A routing x *satisfies the arc capacities* if

$$\sum(x(P) : P \in \mathcal{P}, e \text{ is an arc of } P) \leq u(e) \tag{2}$$

holds for every arc $e \in A$.

We consider several versions of these multicommodity routing problems. In each case, there is a profit $r_i \geq 0$ associated with each demand pair (s_i, t_i) and we wish to maximize

our total profit. The common thread is that we only gain profit for the pair (s_i, t_i) if our routing has *fulfilled* its demand d_i.

We can consider two basic feasibility models: (i) *unsplittable flow* (USF), where we only gain profit for the demand pair (s_i, t_i) if our routing has assigned a flow of value d_i to a single path in $\mathcal{P}_i$, and, (ii) *splittable flow* (SF), where we gain the profit r_i as long as our routing has assigned a total weight of d_i to the paths in $\mathcal{P}_i$. In the splittable version, if we further require our routings to be integral, then we refer to the model as *integral splittable flow* (ISF). Throughout, we let O_u, O_{is}, O_s respectively denote the maximum earnable profit from a feasible routing for the *unsplittable*, *integral-splittable* and *splittable* versions. One natural assumption is made throughout for all versions of the problem:

$$\text{The input graph always has enough capacity}$$
$$\text{to satisfy any single demand.} \tag{3}$$

Note that in the special case when all demands, capacities and profits equal one, both USF and ISF reduce to EDP. Hence these problems are NP-hard themselves, which motivates the investigation of their efficient approximability, which is one of the focuses of this paper.

Bounded Length Edge Disjoint Paths (BLEDP) Problems: A second set of problems that we study here is that of routing a maximum number of edge-disjoint paths between specified source-sink pairs in a network such that each of the paths used is of length at most L, for some length-bound L that is also part of the problem instance. We refer to this problem as BLEDP. As observed in Kleinberg [16], this length constraint, which arises quite naturally in practical routing problems, can transform tractable disjoint paths problems into NP-hard variants. For instance, even the single source-single sink case of this problem, referred to as (s, t)-BLEDP, is NP-hard.

1.1 Overview of Our Results

EDP, USF **and** ISF: Approximation algorithms for USF and EDP have been extensively studied in prior works [23, 10, 1, 18, 19, 15, 16, 17, 28, 2, 21, 22, 6]; [16] provides a comprehensive background on these and related problems. The best known approximation factor for EDP is $O(\sqrt{m})$ [16] (an $O(\sqrt{m})$ approximation for weighted EDP, where profits are not necessarily all one, is presented in [28]). In recent work, an $O(\sqrt{m})$ approximation algorithm has been obtained for the more general USF problem [2], under the assumption that $d_{\max} \leq u_{\min}$.

The preceding result improves upon the $O(\sqrt{m} \log m)$ approximation algorithm of [22] for USF. All these approximation bounds are rather weak and reflect the generally appreciated hardness of these problems. Yet no better hardness than MAX SNP-hardness is known for any of these problems. We prove that even EDP on directed graphs is NP-hard to approximate within a factor of $m^{1/2-\epsilon}$ for any $\epsilon > 0$; our proof is surprisingly simple and does not rely on the PCP theorem. Recently, Ma and Wang [24] have independently shown a weaker hardness result, namely, EDP on directed graphs is *Quasi*-NP-hard to approximate within $2^{O(\log^{1-\epsilon} m)}$ for any $\epsilon > 0$. Their proof uses the hardness of approximating LABEL COVER, and hence relies on the PCP theorem.

On the algorithmic side, we present a simple randomized $O(\sqrt{m} \log^{3/2} m)$ approximation algorithm for USF with polynomially bounded demands, without making the assumption that $d_{\max} \leq u_{\min}$. With the assumption that $d_{\max} \leq u_{\min}$, the approximation guarantee of our algorithm improves to $O(\sqrt{m \log m} \log \log m)$. While the preceding approximation guarantee is weaker than the $O(\sqrt{m})$ bound achieved recently by [2], the significance of our result lies in the fact that our randomized algorithm uses, perhaps, the most basic rounding scheme introduced by Raghavan and Thompson [26] and our analysis relies on elementary combinatorial arguments and straightforward Chernoff-type bounds. We also achieve an $O(\sqrt{m} \log^2 m)$ approximation for USF using a simple greedy algorithm; the main contribution here is extending the analysis of [22] for EDP to handle general capacities. While our strong inapproximability results above apply only for the directed case, we can also prove the (once again tight up to polylogarithmic factors) result that *undirected* USF is NP-hard to approximate within $n^{1/2-\epsilon}$, for any $\epsilon > 0$, if we consider the *node-capacitated* version [17] instead.

The ISF problem was shown to be NP-hard on directed as well as undirected graphs (even with just two sources and sinks) in [7] and also for trees in [10], and to our knowledge no explicit results on its approximability appear in the literature. Our hardness result for EDP trivially implies a similar hardness bound for approximating ISF on directed graphs. In fact, an easy reduction from Independent Set (where the demand pairs play the role of the nodes) shows that the same hardness bound of $m^{1/2-\epsilon}$ applies for the undirected case as well. This same reduction shows that the "fractional" splittable version of the problem remains as hard to approximate. On the algorithmic side, we present a simple greedy algorithm, again generalizing the one in [22], that achieves an approximation guarantee of $O(\sqrt{md_{\max}} \log^2 m)$.

BLEDP: We show that BLEDP can be approximated in polynomial time within a factor of $O(\sqrt{m})$. We prove a matching hardness result of $m^{1/2-\epsilon}$ for any $\epsilon > 0$ that works for undirected graphs as well. For (s,t)-BLEDP we prove an inapproximability result of $m^{1/2-\epsilon}$ for directed graphs, and MAX SNP-hardness for undirected (and directed) graphs. The MAX SNP-hardness applies even when the length bound is a (small) constant. We also present a simple greedy algorithm for BLEDP that achieves an $O(\sqrt{m})$ approximation.

REMARK: In general, directed versions of these problems appear to be harder than their undirected counterparts. Accordingly, all our algorithms are described for the directed case, but they all work for the undirected case as well. Regarding hardness results, unless mentioned otherwise the result applies only to the directed case – but if a hardness result is stated or proved specifically for the undirected case, we stress that a similar result will hold for the directed case as well.

1.2 Organization

A portion of our algorithmic work follows a linear programming (LP) based approach, and hence we begin by describing the relaxations we use, bound their integrality gaps, and note a useful property about the structure of basic feasible solutions in Section 2. In Section 3, we present the hardness results for EDP, USF and ISF, and present an LP-based approximation for USF. We study the hardness of BLEDP problems in Section 4. Finally, in Section 5, we present simple greedy algorithms for all versions of our problems that almost match the corresponding hardness bounds.

2 LP Formulations and Rounding

LP Formulations: A natural relaxation (in the sense of the objective function) of all of the network routing problems that we study in this paper is the following linear program (LP) LP-BASIC.

$$\max\{\sum_{i=1}^{k} \frac{r_i}{d_i}(\sum_{P \in \mathcal{P}_i} x(P)) : \ x \geq 0 \ \text{ and satisfies the}$$

$$\text{constraints } (1),(2)\} \tag{4}$$

Let LP_s denote the optimum value of this LP. Of course, in solving such an LP, one would resort to the well-known compact formulation which uses variables f_e^i for the flow for the

demand pair (s_i, t_i) through edge e, for each $i \in [k]$ and $e \in A$. For the purposes of exposition, however, we view our solutions as vectors in $\mathbf{R}^{\mathcal{P}}$. For any such vector x, we denote by $\mathrm{supp}(x)$ (support of x) the set of paths P for which $x(P) > 0$.

One sees immediately that LP_s may be much more than our desired optimum.

Example 2.1 *Let G be a cycle with a unit capacity on each arc, and a demand of size 2 and a profit of 1 between each pair of vertices. Then $LP_s = m/2$, however, $O_{is} = O_s = 1$ (N.B. $O_u = 0$ and would hence violate (3)).*

For many versions of our problems, we may easily amend the LP formulation to get a tighter relaxation. For instance, in the unsplittable version we define $\mathcal{P}_i^*$ to be those paths in $\mathcal{P}_i$ for which each arc has capacity at least d_i (and let $\mathcal{P}^*$ be the union of these sets). Clearly, any feasible solution to the unsplittable problem may only use the paths in $\mathcal{P}^*$ in its support. Thus we define the linear program LP-UNSPLIT:

$$\max\{\sum_{i=1}^{k} \frac{r_i}{d_i}(\sum_{P \in \mathcal{P}_i^*} x(P)) :\ x \geq 0 \ \text{and satisfies the}$$
$$\text{constraints } (1), (2)\} \tag{5}$$

We use LP_u to denote the optimum value of this LP. Clearly we have $O_u \leq LP_u \leq LP_s$ and $O_u \leq O_{is} \leq O_s \leq LP_s$. Again, we view solutions as vectors in $\mathbf{R}^{\mathcal{P}^*}$. It turns out that this LP is a better approximation for the unsplittable flow problem than (4) was for the splittable flow problem. We will see that $O_u = \Omega(LP_u/(\sqrt{m} \cdot \mathrm{polylog}(m)))$.

For a solution x to any of our LP's, we use $S_g(x)$ to denote the set of (*good*) demand pairs (s_i, t_i) for which (1) is satisfied with equality. Let $S_b(x)$ denotes the set of demand pairs (s_i, t_i) which are not good but for which the left hand side of (1) is positive. Finally, let $S(x) = S_b(x) \cup S_g(x)$ be the set of demand pairs which are at least partially satisfied. There seems to be very little known about the structure of basic solutions for such multi-commodity LP's. One elementary result we can prove is the following.

Proposition 2.1 *If x is a basic optimal solution to (4) or (5), then $|\mathrm{supp}(x)| \leq |S_g(x)| + m$.*

Proof: We argue this proposition for (4) but the same holds for (5). One easily sees that each unit vector in $\mathbf{R}^{\mathcal{P}}$ is feasible for (4) and hence the solution space is full-dimensional (i.e. has dimension $|\mathcal{P}|$). It follows that any basic solution must satisfy some linearly independent subsystem of $|\mathcal{P}|$ constraints. The result now follows. ∎

Proposition 2.1 gives the following weak lower bound on the number of pairs which are satisfied by an unsplit flow (i.e. their demand is satisfied along a single flow path).

Proposition 2.2 *If x is a basic optimal solution to (4), then at least $|S(x)| - m$ pairs are satisfied by an unsplit flow under x.*

Proof: Let x be a basic optimal solution to (4). Each demand pair (s_i, t_i) is satisfied by an unsplit flow under x if and only if $(s_i, t_i) \in S_g(x)$ and $|\text{supp}(x) \cap \mathcal{P}_i| = 1$. Let $S_u(x)$ denote the set of such pairs. Then, by Proposition 2.1, we have:

$$
\begin{aligned}
|S_g(x)| + m &\geq |supp(x)| \\
&\geq |S_u(x)| + |S_b(x)| + 2|S_g(x) - S_u(x)|
\end{aligned}
$$

The last inequality above follows from the fact that each pair in $S_g(x) \setminus S_u(x)$ has at least two flow-paths associated with it. Hence $|S_u(x)| \geq |S_b(x)| + |S_g(x)| - m = |S(x)| - m$, as required. ∎

If our demands set $\mathcal{T}$ consists of every pair of vertices in G and for each pair of vertices $i \neq j$ there is a positive integral demand $d(ij)$, then we call this an *all-pairs* instance. The preceding immediately implies:

Corollary 2.1 *If all profits equal one, i.e if $r \equiv \bar{1}$, then $O_s \geq O_{is} \geq O_u \geq LP_s - m$, and in particular for any all-pairs instance, the LP (4) gives a 2-approximation algorithm as long as $d(ij) \leq u(ij)$ for each $(i, j) \in A$.* ∎

Another corollary, using Example 2.1, is a tightness result for the formulation (4) – the proof is easy and is omitted.

Corollary 2.2 $O_s \geq O_{is} \geq O_u \geq \frac{1}{2m} LP_s$. ∎

Standard Rounding and Deviation Bounds: We will later show how the LP formulation for USF can be rounded to obtain an approximate solution with performance ratio almost matching our hardness result. We now review a standard rounding technique and develop some bounds on its performance that we will use in our analysis later.

Consider a solution to our LP formulation for USF. Let z_i denote the fraction of the demand between the pair (s_i, t_i) that is satisfied by this solution. Decompose the flow of value $z_i d_i$ into a set of flow paths $\{\Gamma_{i,1}, \Gamma_{i,2}, ..., \Gamma_{i,q_i}\}$ where flow on path $\Gamma_{i,j}$ is given by $f_{i,j}$.

Now consider the following randomized rounding procedure, introduced by Raghavan and Thompson [26] in a classic paper. We first choose whether or not to route pair (s_i, t_i); each pair (s_i, t_i) is routed with probability z_i, and the decision is made independent of any other pairs. If we choose to route pair (s_i, t_i), we toss a q_i-sided dice with the property that the jth face shows up with probability $f_{i,j}/(z_i d_i)$. We choose to route the pair along the path $\Gamma_{i,j}$ if the jth face turns up. This rounding procedure is referred to as the *standard rounding* from here on. In what follows, we develop some properties of this rounding procedure. We start by stating some well-known deviation bounds [4, 11].

Proposition 2.3 *(Chernoff-Hoeffding Bounds) Let X_1, X_2, ..., X_ℓ be a set of k independent random variables in $[0, 1]$ and let $X = \sum_{i=1}^{\ell} X_i$.*

1. *For any $\delta \geq 0$, we have:*

$$\Pr[X > (1 + \delta)E(X)] \leq \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^{E(X)}$$

2. *For $0 \leq \delta < 1$, we have:*

$$\Pr[X < (1 - \delta)E(X)] \leq e^{-\delta^2 E(X)/2}.$$

Proposition 2.4 *Consider an LP solution to an instance of the USF problem in which each demand is at least 1. Let S be a set of arcs and let f_e denote the flow through an arc $e \in S$ in an LP solution. Then if X is the random variable indicating the total number of paths in the standard rounding solution that use at least one arc in the set S, $\Pr[X > c \cdot \max\{\sum_{e \in S} f_e, \log m\}] < 1/m^2$ for some suitably large constant c.*

Proof: For each demand pair (s_i, t_i), consider the standard decomposition of the flow from s_i to t_i in the LP solution into at most m path flows from s_i to t_i and a flow of value 0. Let Γ_i denote the set of s_i-t_i paths in the flow decomposition that use at least one arc in S. Let $f(\Gamma_i)$ denote the total flow on all flow paths in the set Γ_i. Finally, let X_i be a 0/1 random variable that indicates whether or not any path in Γ_i is chosen in the rounded solution. Clearly, $E(X_i) = f(\Gamma_i)/d_i$ and $X = \sum_i X_i$. By linearity of expectation, we have $E[X] = \sum_i f(\Gamma_i)/d_i$.

Let c be a positive real constant and let

$$\delta = \frac{c \cdot \max\{\sum_i (f(\Gamma_i)/d_i), \log m\}}{E[X]} - 1.$$

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By the definition of δ, we have $1 + \delta \geq c$. Since X_i's are independent random variables, we can use the bound in Proposition 2.3(1) to derive the following inequality for $c \geq 2e$.

$$\Pr[X > c \cdot \max\{E[X], \log m\}] \quad \leq \quad \left(\frac{1}{2^{1+\delta}}\right)^{E(X)}$$

$$\leq \quad \frac{1}{m^2}.$$

The proposition now follows from the fact that $\sum_{e \in S} f_e \geq \sum_i f(\Gamma_i) \geq \sum_i f(\Gamma_i)/d_i$ since each d_i is at least 1. ∎

Corollary 2.3 *If $u_{\min}/d_{\max} \geq c \log m$ for some suitably large constant c, then* USF *can be approximated to within a constant factor.*

Proof: Solve LP-UNSPLIT, scale the flow down by a factor of $1/c$, where c is a positive constant, and perform the standard rounding. If c is chosen sufficiently large, then it follows from Proposition 2.3(1) that w.h.p.[1] no arc capacity is violated in the rounding procedure. On the other hand, the expected value of the solution obtained is $\Omega(LP_u)$. The result follows. ∎

3 EDP, USF and ISF

3.1 Hardness of Approximating EDP (USF and ISF)

Recall that in the *edge-disjoint paths problem* (EDP), we are given a graph G and a set $\mathcal{T}$ of k source-sink pairs $(s_1, t_1), \ldots, (s_k, t_k)$, and the goal is to find a subset $\mathcal{S} \subseteq \mathcal{T}$ of maximum cardinality such that all (s_i, t_i) pairs in $\mathcal{S}$ can be connected by edge-disjoint paths. The best known approximation factor for EDP is an $O(\sqrt{m})$-approximation [16]. We now prove an essentially matching hardness result on directed graphs.

Theorem 1 *Suppose we are given a directed instance $[G = (V, A), \mathcal{T} = \{(s_i, t_i) : i \in [k], s_i, t_i \in V\}]$ with $|A| = m$. Then, for any $\epsilon > 0$, it is NP-hard to distinguish whether all k pairs in $\mathcal{T}$ can be connected by edge-disjoint paths or at most a fraction $1/m^{1/2-\epsilon}$ of the k pairs can be connected.*

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[1]We use the abbreviation "w.h.p" throughout the paper to mean "with high probability" or, more precisely, "with probability $1 - n^{-c}$, where n is the number of nodes in the graph and c is a constant that can be set arbitrarily large by appropriately adjusting other constants defined within the relevant context."
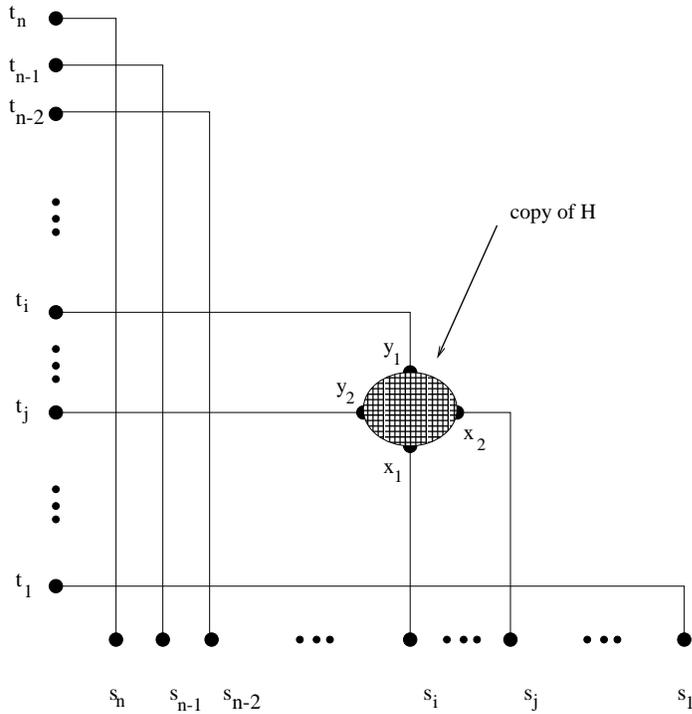
Figure 1: *The reduction from* 2DIRPATH *to* EDP

In other words, the difficulty of the problem is not due to determining which subset of pairs to "route", but lies in determining the edge-disjoint routes for given pairs.

Proof of Theorem 1: The proof is by a reduction from the following well-known NP-hard problem [8]:

Problem: 2DIRPATH:

Instance: A *directed* graph $H = (V, A)$, *distinct* vertices $x_1, x_2, y_1, y_2 \in V$.

Question: Are there two edge-disjoint directed paths, one from x_1 to y_1 and the other from x_2 to y_2, in H?

Given an $\epsilon > 0$, we construct a directed graph G (which will be the directed graph underlying our EDP instance) from H as follows. The starting point in the construction of G will be the graph G', which we refer to as the *skeleton,* whose basic structure is described below. The graph G' will comprise of vertices s_i, t_i for $1 \leq i \leq N$, where $N = |A|^{\lceil 1/\epsilon \rceil}$, together with vertices $\{h_{ij}^{(1)}, h_{ij}^{(2)}, v_{ij}^{(1)}, v_{ij}^{(2)} : 1 \leq j < i \leq N\}$ and the "diagonal" vertices $\{d_{ii} : 1 \leq i \leq N\}$. The vertices are connected in a grid-like fashion as follows. Each s_i is

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connected by a directed path $\mathcal{P}_i$ to t_i where

$$
\begin{aligned}
\mathcal{P}_i \;=\; & \Big[s_i, v_{i1}^{(1)}, v_{i1}^{(2)}, v_{i2}^{(1)}, v_{i2}^{(2)}, \ldots, v_{i,i-1}^{(1)}, v_{i,i-1}^{(2)}, d_{ii}, \\
& \; h_{i+1,i}^{(1)}, h_{i+1,i}^{(2)}, \ldots, h_{n,i}^{(1)}, h_{n,i}^{(2)}, t_i \Big]
\end{aligned}
$$

Note that there are 4 points associated with each grid point except the diagonal vertices and the boundary vertices. The edge set of G' will be the (disjoint) union of the edges in the paths $\mathcal{P}_i$ for $1 \le i \le N$. (For geometric intuition, one can identify s_i, t_i with the points $(N - i + 1, 0)$ and $(0, i)$ on the 2-dimensional grid, and visualize $\mathcal{P}_i$ as the path connecting $(N - i + 1, 0)$ with $(0, i)$ on the 2-dimensional grid that first goes "up" the y-direction, makes a *single bend*, and then goes "left" along the x-direction. The "ij intersection" can be thought of as located at the point $(N - i + 1, j)$ of the 2-dimensional grid.)

The graph G is constructed by making the following modification to G' at each "ij intersection" for $1 \le j < i \le N$: (i) remove the edges $(h_{ij}^{(1)}, h_{ij}^{(2)})$ and $(v_{ij}^{(1)}, v_{ij}^{(2)})$, and (ii) place a copy of the graph H at the ij intersection while identifying the vertices x_1, y_1, x_2, y_2 of the copy of H with the vertices $v_{ij}^{(1)}, v_{ij}^{(2)}, h_{ij}^{(1)}, h_{ij}^{(2)}$, respectively, of the ij intersection. The instance of EDP will now comprise of the graph G together with the N source-sinks pairs $\mathcal{T} = \{(s_i, t_i) : i \in [N]\}$.

Claim 1 *If there are edge-disjoint paths from x_1 to y_1 and x_2 to y_2 in H, then there are N edge-disjoint paths in G, one connecting s_i to t_i, for each $i \in [N]$.*

Proof: Suppose A_1 and A_2 are two edge-disjoint paths in H connecting x_1 to y_1 and x_2 to y_2 respectively. For $r \in [N]$, define the path $\mathcal{Q}_r$ in G to be the s_r-t_r path that is the same as the path $\mathcal{P}_r$ of the skeleton G' except that instead of using the edges $(h_{ij}^{(1)}, h_{ij}^{(2)})$ and $(v_{ij}^{(1)}, v_{ij}^{(2)})$ (for the relevant values of i, j for the path $\mathcal{P}_r$), it uses the edge-disjoint paths A_1 and A_2, respectively, of the local copy of H at the ij intersection. The paths $\mathcal{Q}_r$ thus defined are clearly edge-disjoint. ■

Claim 2 *If there exist edge-disjoint paths $\mathcal{Q}_{i_1}$ and $\mathcal{Q}_{i_2}$ in G connecting s_{i_1} to t_{i_1} and s_{i_2} to t_{i_2}, respectively, for any $1 \le i_1 \ne i_2 \le N$, then there must be two edge-disjoint paths in H from x_1 to y_1 and x_2 to y_2.*

Proof: Identify G with its embedding in the plane. Clearly, one can extend $\mathcal{Q}_{i_1}$ to a closed contour where s_{i_2}, t_{i_2} are on the outside and inside respectively. It follows (cf. [25])

that $\mathcal{Q}_{i_2}$ must cross this contour, but this implies that there exists p, q such that $\mathcal{Q}_{i_1}, \mathcal{Q}_{i_2}$ each uses one of the edges associated with the pq intersection.

Now going back to the graph G, the two paths $\mathcal{Q}_{i_1}$ and $\mathcal{Q}_{i_2}$ will intersect at a point as guaranteed by the above fact, i.e $\mathcal{Q}_{i_1}$ enters at x_1 and leaves at y_1 in the copy of H at this intersection, while $\mathcal{Q}_{i_2}$ enters at x_2 and leaves at y_2, and the edge-disjointness of $\mathcal{Q}_{i_1}$ and $\mathcal{Q}_{i_2}$ implies that there must exist two edge-disjoint paths in H from x_1 to y_1 and x_2 to y_2. ∎

The above two claims imply that YES instances of 2DIRPATH are mapped to instances of EDP where all N pairs can be satisfied, while NO instances are mapped to instances of EDP where at most one pair is satisfied. This creates a gap of N, and since the number of arcs in G equals $m = O(N^2|A|) = O(N^{2+\epsilon})$ (recall that $N = |A|^{\lceil 1/\epsilon \rceil}$), the gap equals $\Omega(m^{1/(2+\epsilon)})$. Finally, note that the choice of $\epsilon > 0$ was arbitrary. This completes the proof of the theorem. ∎

Corollary 3.1 *The* USF *and* ISF *problems on directed graphs are NP-hard to approximate within $m^{1/2-\epsilon}$ for any $\epsilon > 0$.*

Extending these $m^{1/2-\epsilon}$-hardness results for EDP, USF and ISF to undirected graphs pose problems of widely varying degrees of difficulty. For ISF the task is straightforward as there is an easy approximation-preserving reduction from the independent set problem in a graph G. Namely, one creates a new graph G' obtained by creating a non-adjacent copy v' of each node in G. Each edge in G' is assigned a capacity of one, and we include a demand between each v and v' for a flow of size d_v, the degree of v. This together with Håstad's inapproximability result for independent set [12], gives

Fact 3.1 *Unless* NP = ZPP, ISF *on undirected graphs cannot be approximated to within $m^{1/2-\epsilon}$ of the optimum in polynomial time, for any $\epsilon > 0$.*

At the other end, the hardness of undirected EDP remains an interesting open question. Indeed, even the hardness of edge-capacitated USF remains open. We show next an inapproximability bound of $n^{1/2-\epsilon}$, for any $\epsilon > 0$, on node-capacitated USF in undirected graphs.

We start by formally defining the undirected node-capacitated USF problem, denoted Undir-Node-USF. We are given an undirected graph $G = (V, E)$ with positive integral capacities on the **nodes**, and k source-sink pairs (s_i, t_i) with a positive integer demand d_i

and a "profit" r_i for $i \in [k]$ — the objective is to find a subset of the pairs that can be routed feasibly (i.e all node capacities are obeyed) and that maximizes the total profit (as before we only get the profit for a pair (s_i, t_i) when its demand d_i is fully routed on a single path). When all profits are equal, we refer to the version as being "unweighted". Note that unweighted Undir-Node-USF is simply the classical vertex-disjoint paths problem when all demands and capacities are equal to 1.

Theorem 2 *It is NP-hard to approximate unweighted* Undir-Node-USF *within a factor of* $n^{1/2-\epsilon}$ *for any* $\epsilon > 0$.

We will see that the above theorem is nearly tight since LP-based (or greedy) algorithms, such as those presented in the following section for the edge-capacity version, can achieve an approximation ratio of $O(\sqrt{n}\,\text{poly}(\log n))$ for Undir-Node-USF.

The proof of Theorem 2 is based on a reduction from SAT along the lines of the NP-hardness proof for two-commodity integral flow presented in [7].

Theorem 3 *Given an instance of* Undir-Node-USF *with two source-sink pairs, it is NP-hard to decide if both pairs can be feasibly routed, even if all node capacities are 1 or 2, and the two demands are 1 and 2.*

Proof: Given an instance of SAT, we will create an undirected graph $G = (V, E)$ with four distinct nodes s_1, s_2, t_1, t_2 and node capacities $c : V \to \{1, 2\}$ with the property that the instance of SAT is a yes-instance if and only if G contains a pair of node-disjoint paths P_1, P_2 such that P_i is a path from s_i to t_i and P_2 only uses nodes of capacity two.

The graph G is obtained by stringing together n cycle gadgets, one for each variable x_i in the SAT instance. Cycle i is formed by combining two paths T_{x_i} and $T_{\bar{x}_i}$ each of which alternates between capacity 2 and capacity 1 nodes. We join the left end-points of these paths into a new node a_i and the right end-points into a new node b_i; each of a_i and b_i have capacity 1. We also create two capacity 2 nodes $C_i, \bar{C}_i$ for each clause i, $1 \le i \le m$, and connect $\bar{C}_i$ to C_{i+1} for $1 \le i < m$, and also include edges from these to nodes in cycle gadgets corresponding to the literals in clause i — see Figure 2. In addition we add vertices s_1, s_2, t_1, t_2 to G and join s_1 (resp. t_1) to the node a_1 in the cycle T_{x_1} (resp. node b_n in T_{x_n}) where x_1 (x_n) is the first (last) variable in the SAT instance as per some arbitrary ordering. The vertex s_2 is connected to C_1 and t_2 to $\bar{C}_m$.
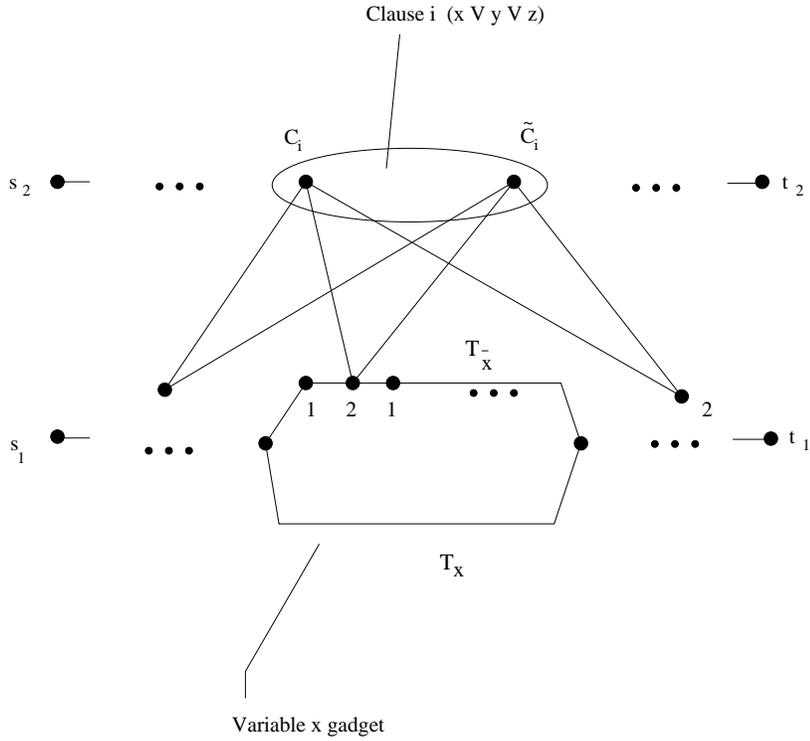
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Figure 2: *The reduction from* SAT *to* Undir-Node-USF

Now consider a pair of paths P_1, P_2 satisfying our Undir-Node-USF problem. The path P_2 must have the form $s_2 C_1 v_1 \bar{C}_1 C_2 v_2 \bar{C}_2 \ldots \bar{C}_m t_2$, where each v_i is a node corresponding to some literal x in clause i. The path P_1 must therefore have the form $s_1 Q_1 Q_2 \ldots Q_n t_1$ where each Q_j is either T_{x_j} or $T_{\bar{x}_j}$. Since each Q_j is disjoint from each v_i, the path P_1 determines a satisfying truth assignment for the instance of SAT in the natural way. Conversely, a solution to the node-capacitated unsplittable flow problem can be obtained from a satisfying truth assignment. ∎

We remark that the preceding proof (and hence Theorem 2) could be extended to showing hardness of USF with only the extra condition that flow paths must be node-disjoint (and not arbitrary node capacities). We now return to our task of proving hardness of Undir-Node-USF. In the following, we refer to an instance of Undir-Node-USF in which there are two source-sink pairs, all node capacities are 1 or 2, and the demands of the two pairs are 1 and 2, as 2PAIR-Undir-Node-USF.

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Proof of Theorem 2: We use a construction similar to the one used in Theorem 1 – the main difference is that at each "ij intersection" we now place a hard instance $\mathcal{I}$ of 2PAIR-Undir-Node-USF above, instead of the 2DIRPATH instance we used in Theorem 1. Suppose the two source-destination pairs in instance $\mathcal{I}$ are (x_1, y_1) and (x_2, y_2), with demands 1 and 2, respectively. We construct the graph for Undir-Node-USF as follows. We start with the same skeleton graph G' of Theorem 1 and make the following modifications. At each ij intersection, $1 \leq j < i \leq N$, we remove the edges $(h_{ij}^{(1)}, h_{ij}^{(2)})$ and $(v_{ij}^{(1)}, v_{ij}^{(2)})$, and place a copy of the instance $\mathcal{I}$ at the ij intersection while identifying the vertices x_1, y_1, x_2, y_2 of the copy of H with the vertices $v_{ij}^{(1)}, v_{ij}^{(2)}, h_{ij}^{(1)}, h_{ij}^{(2)}$, respectively, of the ij intersection. For the nodes in the copy of $\mathcal{I}$ placed at the "ij intersection" (for $1 \leq j < i \leq N$) we change the capacities of nodes with capacity 1 (recall that all nodes in instance $\mathcal{I}$ have capacity 1 or 2) to $n - i$, and those of nodes with capacity 2 to $n - j$. We set the capacity of the vertices $v_{ij}^{(1)}$ and $v_{ij}^{(2)}$ as $n - i$ and the capacity of $h_{ij}^{(1)}$ and $h_{ij}^{(2)}$ as $n - j$. Finally, we set the demand of the pair (s_i, t_i) to $n - i$, for $i \in [N]$.

We first show that if the instance $\mathcal{I}$ is feasible, then the demands of all the N (s_i, t_i) pairs can be fully met by a feasible routing. The argument is along the lines of the proof of Claim 1. Suppose A_1 and A_2 are two node-disjoint paths in $\mathcal{I}$ such that A_i is a path from x_i to y_i, $i \in \{1, 2\}$ and A_2 uses only nodes of capacity two. This implies that at any ij intersection in G, $1 \leq j < i \leq N$, there are two node-disjoint paths, one from $v_{ij}^{(1)}$ to $v_{ij}^{(2)}$ that passes through nodes with capacity at least $n - i$, and the other from $h_{ij}^{(1)}$ to $h_{ij}^{(2)}$ that passes through nodes with capacity at least $n - j$. For each i, define the path Q_i in G to be the s_i-t_i path the same as the single bend path from s_i to t_i in G' except that instead of using the edges $(h_{ij}^{(1)}, h_{ij}^{(2)})$ and $(v_{ij}^{(1)}, v_{ij}^{(2)})$ (for the relevant values of i, j for the path $\mathcal{P}_r$), it uses the node-disjoint paths A_1 and A_2, respectively, of the local copy of $\mathcal{I}$ at the ij intersection. The set of paths Q_i are node-disjoint and at each ij intersection, the path Q_i passes through nodes with capacity at least $n - i$, thus completing our first claim.

We now show that if there exist two paths Q_{i_1} and Q_{i_2} connecting s_{i_1} to t_{i_1} and s_{i_2} to t_{i_2}, respectively, for any $1 \leq i_1 < i_2 \leq N$, then there must be two node-disjoint paths in $\mathcal{I}$, one from x_1 to y_1 and from x_2 to y_2, with the latter path traversing nodes of capacity of 2 only. Since the capacity of the vertices $v_{ij}^{(1)}$ and $v_{ij}^{(2)}$, for $i > i_1$, is less than $n - i_1$, the path Q_{i_1} (resp., Q_{i_2}) cannot pass through any vertex $v_{ij}^{(1)}$ or $v_{ij}^{(2)}$, for $i > i_1$ (resp., for $i > i_2$). Similarly, since the capacity of the vertices $h_{ij}^{(1)}$ and $h_{ij}^{(2)}$, for $j > i_1$, is less than $n - i_1$, Q_{i_1} cannot pass through any vertex $h_{ij}^{(1)}$ or $h_{ij}^{(2)}$, for $j > i_1$. It thus follows that

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the path Q_{i_1} passes through each ii_1 intersection, $i > i_1$. A geometric argument along the lines of Claim 2 implies that Q_{i_2} intersects with Q_{i_1} at some ii_1 intersection. Due to the above restriction on path Q_{i_2}, it follows that the two paths meet at some ii_1 intersection, for $i_1 < i \leq i_2$. The subgraph placed at the ii_1 intersection has node capacity either $n - i_1$ or $n - i$. Since the demand being routed along path Q_{i_1} is $n - i_1 > n - i$, it follows that the path Q_{i_1} uses nodes of capacity exclusively $n - i_1$. By construction, the subpaths of Q_{i_1} and Q_{i_2} in the ii_1 intersection yield two paths A_2 and A_1, respectively, for the instance $\mathcal{I}$, that satisfy the following conditions: The path A_1 is from x_1 to y_1, the path A_2 is from x_2 to y_2 and passes through nodes of capacity 2 only, and the two paths are node-disjoint. This completes the proof of our second claim.

Analogous to the proof of Theorem 1, the above two claims establish that it is NP-hard to approximate unweighted Undir-Node-USF to within a factor of $O(n^{1/2-\epsilon})$, where n is the number of nodes in the Undir-Node-USF instance. ∎

In the next section we give an (essentially tight) approximation algorithm for USF based on rounding an optimal solution to the LP (5). Standard rounding techniques are not as easily applied to ISF; we defer the positive results for the latter problem to Section 5.

3.2 An LP-Based Approximation for USF

We now give a simple algorithm to approximate USF to within $O(\sqrt{m \log m} \log \log m)$ when $u_{\min} \geq d_{\max}$, and an $O(\sqrt{m}(\log m)^{3/2})$-approximation without this assumption, but with the requirement that $d_{\max}$ is polynomially bounded.

Theorem 4 *If all demands are between 1 and 2, then there is a polynomial-time randomized algorithm to compute a routing for* USF *that, w.h.p, achieves an approximation factor of* $O(\sqrt{m \log m})$ *and satisfies all capacity constraints.*

Proof: We assume w.l.o.g. that the maximum profit, $\max_i r_i$, is 1; otherwise, we can scale the profits appropriately. We first solve LP-UNSPLIT. If the LP value LP_u is less than $\sqrt{m}$, then routing the demand with the highest profit yields a $\sqrt{m}$ approximation. Therefore, in the remainder of the proof, we assume that $LP_u \geq \sqrt{m}$. We scale down the LP flow by a factor of $1/6$, and perform standard rounding. In the rounding procedure, each demand is selected independently with a probability equal to one-sixth of the fraction routed in the LP. Thus, the expected total profit of the rounded solution is $\Omega(LP_u)$. Since

$LP_u \geq \sqrt{m}$ and the profit of each demand is in $[0, 1]$, it follows from Proposition 2.3(2) that w.h.p. the total profit of the rounded solution is $\Omega(LP_u)$. The rounded solution, however, may violate the capacity of many arcs. We now show that we can obtain a solution which w.h.p. has value $\Omega(LP_u/\sqrt{m \log m})$ and does not violate any arc capacities.

HEAVY AND LIGHT EDGES. Let c_0 denote the quantity $12 \ln m$. Call an arc x *heavy* if $u(x) > c_0$ and *light* otherwise. Our first claim is that with probability at least $(1 - 1/m)$, the rounded solution does not violate the capacity of a heavy arc. To see this, define X_x to be a random variable that indicates the *total number of demands* routed through an arc x. Therefore, $E[X_x] \leq u(x)/6$. Applying Proposition 2.3(1) with $\delta = u(x)/(2E(X_x)) - 1$, we obtain

$$\begin{aligned}
\Pr[X_x > \frac{u(x)}{2}] &\leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}} \right)^{E[X_x]} \\
&\leq \left(\frac{e}{u(e)/(2E(X_x))} \right)^{E[X_x]} \\
&\leq \left(\frac{e}{3} \right)^{6 \ln m} \\
&\leq \frac{1}{m}.
\end{aligned}$$

In the third step, we use the inequality $E[X_x] \leq u(e)/6$, and in the final step, we use $12 \ln m$.

We have thus shown that with probability at least $1 - 1/m$, at most $u(x)/2$ paths are chosen to go through an arc x of capacity at least $12 \ln m$. Since all demands are between 1 and 2, it follows that, with probability at least $1 - 1/m$, no heavy arc is violated.

PROCESSING THE ROUNDED SOLUTION. Now let $\mathcal{R}$ denote the set of paths in a solution obtained by standard rounding. A path $P \in \mathcal{R}$ is said to be α-*light* if the total capacity associated with the light arcs that lie on this path is exactly α. Partition $\mathcal{R}$ into two sets $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $\mathcal{R}_1$ is the set of all α-light paths with $\alpha \geq \sqrt{mc_0}$, and $\mathcal{R}_2 = \mathcal{R} \setminus \mathcal{R}_1$. It is easy to see that $\mathcal{R}$ contains at most mc_0/α paths which are α-light. Thus, $|\mathcal{R}_1|$ is at most $\sqrt{mc_0}$.

Initialize S_1 to be the single-element set consisting of a path of largest profit in $\mathcal{R}_1$. For a given set S of paths, let PROFIT(S) denote the sum of the profits of the source-sink pairs routed in S. Clearly, PROFIT(S_1) $\geq$ PROFIT($\mathcal{R}_1$)$/\sqrt{mc_0}$. Construct also another set S_2 of paths as follows: Repeatedly pick a path P of largest profit from $\mathcal{R}_2$, add P to S_2, and delete P along with all paths in $\mathcal{R}_2$ that share a light arc with P. Since the total capacity of light arcs is at most $\sqrt{mc_0}$, it follows from Proposition 2.4 that for any path

P in $\mathcal{R}_2$, the total number of paths in $\mathcal{R}$ that share a light arc with P is $O(\sqrt{mc_0})$ with probability at least $1 - 1/m^2$. Therefore, with probability at least $1 - 1/m$, for every path added to S_2, only $O(\sqrt{mc_0})$ other paths in $\mathcal{R}_2$ are thrown away. Thus, upon termination (i.e. when $\mathcal{R}_2 = \emptyset$), $\text{PROFIT}(S_2)$ is $\Omega(\text{PROFIT}(\mathcal{R}_2)/\sqrt{mc_0})$ w.h.p. Output the better among the solutions S_1 and S_2.

APPROXIMATION RATIO AND CORRECTNESS. It is clear from the preceding description that we are guaranteed a profit of $\Omega(\text{PROFIT}(\mathcal{R})/\sqrt{mc_0})$; this yields the claimed approximation guarantee as $\text{PROFIT}(\mathcal{R})$ is $\Omega(LP_u)$ w.h.p. To see that with w.h.p., the set of paths in S_2 does not violate the capacity of any arc, observe that: (a) no two paths in S_2 share a light arc, and (b) with probability $1 - 1/m$, the set of paths in $\mathcal{R}$ does not violate the capacity of a heavy arc. ■

Theorem 5 *If all demands are polynomially bounded, then there is a polynomial-time randomized algorithm to compute a routing for* USF *that, w.h.p, achieves an approximation factor of $O(\sqrt{m}\log^{3/2} m)$ and satisfies all capacity constraints.*

Proof: Let $d_{\max} = O(m^{O(1)})$ denote the largest demand. Partition the demands into $O(\log m)$ classes, say $D_0, D_1, ...$ where the class D_i contains all demands in the interval $[2^i, 2^{i+1})$. Find a solution for each demand class using the approach of Theorem 4. Output the solution with the largest profit. By pigeonhole principle, one of these classes contains $\Omega(1/\log m)$-fraction of the optimum solution's total profit. The theorem follows. ■

Theorem 6 USF *with arbitrary demands can be approximated to within a $O(\sqrt{m\log m}\log\log m)$ factor if the minimum arc capacity is at least as large as the largest demand.*

Proof: Without loss of generality, we may assume that $d_{min} = 1$. As before, partition the demands into classes $D_0, D_1, ..., D_q$ where the class D_i contains all demands in the interval $[2^i, 2^{i+1})$. Set $p = q - (\log\log m + \log c)$ where c is the constant of Corollary 2.3, and define $\mathcal{D}_1 = \bigcup_{i=1}^{p} D_i$ and $\mathcal{D}_2 = \bigcup_{i=p}^{q} D_i$. Also, for some optimal solution, let OPT_1 and OPT_2 denote the profit generated by commodities in the demand sets $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively.

Since the maximum demand in $\mathcal{D}_1$ is at most

$$(d_{max}/c\log m) \leq (u_{min}/c\log m),$$

by Corollary 2.3, it follows that OPT_1 can be approximated to within a constant factor. On the other hand, we can use Theorem 4 to find an approximate solution for each demand class

D_i with $p \leq i \leq q$. By pigeonhole principle, one of these classes contains $\Omega(1/\log\log m)$-fraction of OPT_2. Choosing the better of the two solutions, we get the desired approximation ratio. ∎

4 Bounded Length Edge-Disjoint Paths

We are given a (possibly directed) graph $G = (V, E)$, in which each edge e has a given nonnegative length. We are also given an integer L, which is referred to as the *length bound*. The length of a path is defined to be the sum of the lengths of the edges on the path. In BLEDP, we are given a set $\mathcal{T}$ of k source-sink pairs $(s_1, t_1), \ldots, (s_k, t_k)$, and the goal is to find a subset $\mathcal{S} \subseteq \mathcal{T}$ of maximum cardinality such that all (s_i, t_i) pairs in $\mathcal{S}$ can be connected by edge-disjoint paths, each path of length at most L. In the (s, t)-BLEDP problem, we are given a *single* pair (s, t), the goal is to find a maximum number of edge-disjoint s–t paths where each path has length at most L.

We consider (α, β)-approximations for the BLEDP problems. An (α, β)-approximation algorithm is one that obtains for each instance $\mathcal{I}$ with length bound L, $\alpha \cdot \text{OPT}(\mathcal{I})$ edge-disjoint paths each of length at most βL, where $\text{OPT}(\mathcal{I})$ is the size of the optimal solution. For convenience, we refer to an $(\alpha, 1)$-approximation as an α-approximation.

4.1 (α, β)-Approximating (s, t)-BLEDP

4.1.1 Inapproximability results

A simple reduction from 2DIRPATH shows that the directed version of (s, t)-BLEDP does not have a polynomial time $(\frac{1}{2} + \epsilon, \frac{4}{3} - \epsilon)$-approximation algorithm unless P = NP. This reduction, however, is inexorably tied to the directed case; indeed the undirected version of 2DIRPATH is solvable in polynomial time by the work of Robertson and Seymour [27]. One could possibly attack the undirected case, however, by considering the related NP-hard problem (see [7]) INTEGER2COMMODITY where in we are given an *undirected* graph $G = (V, E)$ and *distinct* vertices $x_1, x_2, y_1, y_2 \in V$, and the objective is to find a maximum collection of edge-disjoint paths, each joining x_i to y_i for $i = 1$ or 2. We note that the half-integral version of INTEGER2COMMODITY flow was shown to be polynomially solvable in a classical paper of Hu [13]. We now give a reduction which shows MAX SNP-hardness of both undirected (s, t)-BLEDP and INTEGER2COMMODITY; a similar result for undirected

(s, t)-BLEDP has recently been obtained independently in [3].

Theorem 7 *For both undirected as well as directed graphs, (s, t)-BLEDP with a length bound of six is MAX SNP-hard. Hence, there exist constants α and β, $\alpha < 1$ and $\beta > 1$, such that there is no polynomial-time (α, β)-approximation for (s, t)-BLEDP unless P=NP.*

Proof: We prove the desired claim for undirected graphs only. The proof can be easily amended to apply to directed graphs.

The reduction is from Bounded Occurrence 3-Dimensional Matching (3DM) and follows the reduction presented in [21] which in turn is motivated by ideas from [10]. In an instance $\mathcal{I}$ of 3DM, we are given three disjoint sets $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_n\}$ and $C = \{c_1, c_2, \ldots, c_n\}$, and a set $\mathcal{T}$ of m triples $T_\mu \in A \times B \times C$, $\mu \in [m]$. It is shown in [14] that there exists an $\epsilon_0 > 0$ such it is NP-hard to distinguish between instances $\mathcal{I}$ where there exist n *disjoint* triples in $\mathcal{T}$ (call them "satisfiable" instances) and those where there are at most $(1 - \epsilon_0)n$ disjoint triples in $\mathcal{T}$ (call these "ϵ_0-unsatisfied" instances), even if we assume that each element of A, B, C is in the same constant number M of triples in $\mathcal{T}$.[2] We denote the μth triple T_μ as $(a_{p_\mu}, b_{q_\mu}, c_{r_\mu})$ for some $1 \le p_\mu, q_\mu, r_\mu \le n$.

Create an undirected graph $H = (V, E)$ as follows:

$$
\begin{aligned}
V \;=\; & \{b_i, c_i : i \in [n]\} \bigcup \{x_\mu, y_\mu : \mu \in [m]\} \\
& \bigcup \{a_{il} : i \in [n], l \in [M-1]\} \bigcup \{s, t\} \\
E \;=\; & \{(s, b_i), (c_i, t), (a_{il}, t) : i \in [n], l \in [M-1]\} \\
& \bigcup \{(s, x_\mu), (y_\mu, a_{p_\mu, l}) : \mu \in [m], l \in [M-1]\} \\
& \bigcup \{(b_{q_\mu}, x_\mu), (x_\mu, y_\mu), (y_\mu, c_{r_\mu}) : \mu \in [m]\}
\end{aligned}
$$

The graph H contains a vertex for each element in the sets B and C, and $(M-1)$ vertices for each element in A. In addition, there are 2 special vertices s and t, and tuple vertices x_μ and y_μ, for each triple μ. The vertex s is connected to each vertex corresponding to elements in B, and the tuple vertices $x_1, x_2, ..., x_m$. Similarly, the vertex t is connected to each vertex corresponding to elements in C, and the $n(M-1)$ vertices corresponding to the elements in A. Finally, for each tuple $T_\mu = (a_{p_\mu}, b_{q_\mu}, c_{r_\mu})$, we connect x_μ to b_{q_μ} and y_μ, y_μ to c_{r_μ} as well as $(M-1)$ copies of a_{p_μ}. We assign positive integer lengths on every edge

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[2] Actually we only need the fact that each element in A occurs in some constant number of triples.

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of H and consider the problem of finding the maximum number of edge-disjoint s-t paths of total length at most 6 in H. The edges (s, x_μ) for $\mu \in [m]$ have a "length" of 3 each, and the edges (y_μ, c_{r_μ}) have a length of 2, while all other edges have a length equal to 1.

We now claim that (a) [completeness] if the instance $\mathcal{I}$ of 3DM is *satisfiable*, then we can find Mn edge-disjoint paths from s to t of length at most 6, and (b) [soundness] if $\mathcal{I}$ is ϵ_0-unsatisfied, then there are at most $(M - \epsilon_0/2)n$ such s-t paths in H.

Let us first verify the completeness. Suppose $T_{\mu_1}, T_{\mu_2}, \ldots, T_{\mu_n}$ are n *disjoint* triples. Denote $f_i = q_{\mu_i}$ and $g_i = r_{\mu_i}$ for $i \in [n]$. Define the paths $P_i = [s, b_{f_i}, x_{\mu_i}, y_{\mu_i}, c_{g_i}, t]$ for $i \in [n]$, and the paths $Q_\mu = [s, x_\mu, y_\mu, a_{p_\mu, l_\mu}, t]$ for $\mu \in [m] - \{\mu_1, \mu_2, \ldots, \mu_n\}$ and an $l_\mu \in [M - 1]$ defined such that no two μ's whose corresponding triples "share" an element of A, have the same value of l_μ. (This is possible to achieve since the fact that each element of A is present in exactly one triple among $T_{\mu_1}, \ldots, T_{\mu_n}$ implies that, for each element $a_i \in A$, there are $M - 1$ such μ's in the set $[m] - \{\mu_1, \ldots, \mu_n\}$.) It is now easy to check that the paths P_i and Q_μ are all edge-disjoint and that each has length exactly 6.

To prove soundness, suppose there is a collection $\mathcal{C}$ of $(Mn - \delta n)$ edge-disjoint s-t paths of length at most 6. Clearly at most $(M - 1)n$ of these paths use a final edge of the form $a_{p_\mu, l} t$ and hence at least $(1 - \delta)n$ of the paths in $\mathcal{C}$ use a final edge of type $c_i t$. Any such path of length at most 6 is easily verified to be of the form P_i above. The b_i's and c_j's used in these paths are all distinct and yield a set $\mathcal{S}$ of $(1 - \delta)n$ triples which have distinct B and C "coordinates". If some a_i is present in say $t > 1$ triples in $\mathcal{S}$, we can only take 1 of these in a 3-matching. But then we also "lost" at least $t - 1$ potential paths of the form Q_μ for our collection $\mathcal{C}$. Thus a_i was used to get at most $(M - t)$ paths of the form Q_μ (as opposed to the $M - 1$ paths in the completeness case). We can clearly lose at most δn such paths in all, and hence by retaining in $\mathcal{S}$ at most one triple containing any a_i, for each $i \in [n]$, we will be left with a set of at least $(1 - 2\delta)n$ **disjoint** triples.

We have thus proved that the undirected (s, t)-BLEDP problem (even with a length bound of 6) is MAX SNP-hard, and in fact for any $\epsilon > 0$, there is no polynomial-time $(1 - \frac{\epsilon_0}{2M}, \frac{7}{6} - \epsilon)$-approximation for (s, t)-BLEDP unless P=NP. ∎

One can easily amend the previous reduction to obtain the following claim.

Corollary 4.1 INTEGER2COMMODITY *is MAX SNP-hard.*

Proof: We create an instance of INTEGER2COMMODITY by taking the graph H in the preceding proof, and splitting s and t into s_1, s_2, t_1, t_2 as follows. The node s_1 will be

adjacent only to the nodes b_i and t_1 will be adjacent only to the c_i. Similarly, s_2 is adjacent to the x_μ's and t_2 is adjacent only to the a_{il}'s. The $s_i - t_i$ paths are one of the two types prescribed in the original proof. ∎

4.1.2 An $(1 - \epsilon, 1/\epsilon)$-approximation algorithm for (s, t)-BLEDP

We are given an undirected graph G, length bound L, a source s and a sink t in G. Let p be the optimal number of edge-disjoint paths from s to t, where each path has length at most L. In this section, we present a simple algorithm that obtains, for any positive real ϵ, at least $(1 - \epsilon)k$ paths, each of length at most L/ϵ.

We first define a minimum-cost flow problem in G from s to t, where the cost of an edge is defined to be its length and the value of the desired flow is p. Since there exists a flow of value p and cost at most pL, the minimum-cost flow algorithm returns a set S of edge-disjoint paths from s to t such that the sum of all of the path lengths is at most pL. By averaging, it follows that at least $(1 - \epsilon)p$ s-t paths in S have length at most L/ϵ, thus yielding the desired approximation.

4.2 Hardness of α-approximating BLEDP

4.2.1 Hardness of (s, t)-BLEDP

Theorem 8 *For directed graphs, (s, t)-BLEDP is NP-hard to approximate within a factor of $m^{1/2 - \epsilon}$ for any $\epsilon > 0$.*

Proof: The reduction is from 2DIRPATH. Given an instance $[H; x_1, x_2, y_1, y_2]$ of 2DIRPATH, we construct an instance $[G = (V, A'); s, t; L]$ where G is the same directed graph as the one constructed in the proof of Theorem 1 together with two new vertices s and t, and arcs joining s to $s_1, s_2, \ldots, s_N$ and joining $t_1, t_2, \ldots, t_N$ to t, and L is a suitable length bound to be mentioned later in the proof.

We now define non-negative integer lengths on the arcs and consider the problem of finding edge-disjoint paths of total length at most L between s and t. In the directed graph G, the arcs at the bend (the ones incident to d_{ii} for $i \in [N]$) and the arcs which lie wholly within the various copies of H, get a length of zero. The arc (s, s_i) gets a length of i, and the arc (t_i, t) gets a length of $N - i$, for $i \in [N]$, and all other arcs have a length of 1.

We now claim that any s, t path P with total length at most $2N - 1$ must use a "bend", i.e must go through d_{rr} for some $r \in [N]$. Indeed, let the path P go through s_i and t_j.

Then, if it does not use a bend, all arcs it uses in the underlying grid of (the skeleton of) G have a length 1, and at least j such "vertical" arcs and $N - i + 1$ such "horizontal" arcs are required for any directed path through s_i and t_j. This, together with the lengths i and $N - j$ on (s, s_i) and (t_j, t) respectively, implies that P has total length at least $i + (N - j) + j + (N - i + 1) = 2N + 1$. Hence, using the same *geometric argument* as in the proof of Theorem 1, we can prove that in the case when there are no edge-disjoint paths between x_1, y_1 and x_2, y_2 in H, the maximum number of edge-disjoint (s, t)-paths in G of length at most $2N - 1$ is one.

Also, when we start from a YES instance of 2DIRPATH, there are N edge-disjoint (s, t)-directed paths of length $2N - 1$ in G, namely the N paths $\mathcal{P}_i$ where $\mathcal{P}_i$ is the (s, t) path that goes through s_i and t_i via the "bend" vertex d_{ii}. That is, the path $\mathcal{P}_\rangle$ follows i vertical arcs from s_i up to d_{ii} and then $N - i + 1$ horizontal arcs to t_i, for an overall length of $i + (i - 1) + (N - i) + (N - i) = 2N - 1$.

Thus the gap between the optimum values of the instances created from the YES and NO instances of 2DIRPATH is N, which as before will prove an inapproximability bound of $m^{1/2 - \epsilon}$ by choosing N suitably. ∎

4.2.2 Hardness of BLEDP

We now employ an approximation preserving reduction from the independent set problem to establish that BLEDP, even on undirected graphs, is hard to approximate within $m^{1/2 - \epsilon}$.

Theorem 9 *Unless* NP = ZPP, BLEDP *on undirected graphs cannot be approximated in polynomial time within a factor of $m^{1/2 - \epsilon}$ for any $\epsilon > 0$.*

Proof: The reduction is from the independent set problem. Assume we are given a graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$. It is known that unless NP = ZPP, it is not possible to distinguish in polynomial time between the cases when $\alpha(G) \geq n^{1 - \epsilon_0}$ and when $\alpha(G) \leq n^{\epsilon_0}$ for any fixed $\epsilon_0 > 0$ [12].

Starting from G, we construct a "grid graph" instance of BLEDP $[H; \{(s_i, t_i) : i \in [n]\}]$ as follows. The n sources s_i correspond to the vertices $(i, 0)$ respectively, while the n sinks t_i correspond to $(0, n - i + 1)$ respectively. Each s_i is connected to t_i through a "canonical" path $\mathcal{Q}_i = [s_i, a[i, 1], a[i, 2], \ldots, a[i, n-i], b[i-1, n-i+1], b[i-2, n-i+1], \ldots, b[1, n-i+1], t_i]$; here, we view each $a[i, j]$ and $b[i, j]$ is as *an edge* (with a designated entry and exit point) placed at vertex (i, j) of the plane grid (for $1 \leq i, j \leq n$ with $i + j \leq n$). The adjacency

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information of G is encoded in H by *identifying* the edges $a[i, n-j+1]$ and $b[i, n-j+1]$ whenever vertices i and j (with $i < j$) are adjacent in G; this serves the purpose that whenever i and j are adjacent in G, the canonical paths $\mathcal{Q}_i$ and $\mathcal{Q}_j$ will share an edge (viz. the "identified" copy of $a[i, n-j+1]$ and $b[i, n-j+1]$), and thus cannot both be simultaneously routed.

We assign a length of 2 to the following edges: (i) $(s_i, a[i, 1])$ for $1 \leq i < n$, and $(b[1, n-i+1], t_i)$ for $1 < i \leq n$; (ii) $a[i, j]$ and $b[i, j]$ for i, j such that $i + j \leq n$; and (iii) edges joining the "exit" of $a[i, j]$ (respectively $b[i, j]$) to the "entry" of $a[i, j+1]$ (respectively $b[i-1, j]$) for i, j such that $i > 1$ and $i + j < n$. We assign a length of 1 to all other edges.

We now claim that the only s_i-t_i path in H of total length at most $4n - 3$ is the canonical path $\mathcal{Q}_i$. We first note that the unique shortest path between any two vertices in the same row (or the same column) is simply the shortest-hop path that is contained in the row (or column). Therefore, it follows that the shortest $s_i - t_i$ path lies wholly within the portion of the grid $X = \{(p, q) : p \leq i \text{ and } q \leq n-i+1\}$, since any other path will contain a subpath connecting two vertices in the same row (or column) that is not the shortest path between the two vertices. Any s_i-t_i path in X passes through at least n grid points. At each grid point, except the point $(i, n-i+1)$, there is an edge of length 2. The edge at $(i, n-i+1)$ has length 1. In addition, there are $n - 1$ edges of length 2 that connect these grid points together. So any path that does not pass through the edge located at grid point $(i, n-i+1)$ has cost at least $4n - 2$. The shortest $s_i - t_i$ path is a path that lies wholly in X, passes through $(i, n-i+1)$, and passes through the least number of other grid points in X. The unique path that satisfies the preceding conditions is $\mathcal{Q}_i$ and has length $4n - 3$.

Thus, starting from G, we can, in polynomial time, construct a graph H with n source-sink pairs (s_i, t_i) such that the maximum number of pairs which can be joined by edge-disjoint paths of length at most $4n - 3$, equals $\alpha(G)$. The gap of $n^{1-2\epsilon_0}$ in $\alpha(G)$ thus clearly translates into a gap of $m^{1/2-\epsilon_0}$ for the BLEDP instance we create, and the proof is completed. ∎

5   Greedy Algorithms for USF, ISF and BLEDP

We now show that the hardness bounds for all problems considered can be matched (up to polylogarithmic factors) by suitable adaptations of a greedy algorithm. We give the "core" algorithm in Figure 3, although subroutines must be tailored for the different problems

considered. In the code description, a network refers to a directed graph together with integer capacities on the arcs. The operation of *decrementation* of a network by a flow f, results in a new network for which the capacity on an arc e is precisely f_e units less than before.

```
GREEDY MULTICOMMODITY ROUTING (GREED)
{
/* Input is a directed network in which each arc e has
length l_e in {0, 1} */
1.   S = ∅;
2.   While (there is an unsatisfied demand)
3.        For each demand pair s-t in D − S
4.             Find a shortest flow f from s to t in the residual network;
                  (i.e., f is an s-t flow that minimizes the measure ∑_e f_e l_e)
5.        EndFor
6.        If no routings are found, then Output Current Solution;
7.        Let s-t be a demand pair that has the shortest flow value,
             among all demand pairs, and let f be the flow from s to t;
8.        Decrement the residual network by the flow f;
9.        Add s to S;
10. EndWhile
     Output Current Solution;
}
```

Figure 3: The "core" greedy algorithm

The flows (or routings) f found by the greedy algorithm (refered to as GREED) depend (e.g., split or unsplit) on the problem version being addressed. In any case, "shortest" (see line 4 of the algorithm) will always refer to minimizing the measure $\sum_e f_e l_e$ where l_e is either 0 or 1. In the cases of USF and ISF, the problem of finding such an f reduces to solving a shortest path or mincost flow problem respectively. For BLEDP with length bound L, the "shortest flow" f is a source-sink path with the fewest number of edges among all of the source-sink paths of length L. In the following, we briefly describe a simple subroutine

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that computes such a path. It is sufficient to provide an algorithm that given a directed graph $D = (V, A)$, a source s, a sink t, and a positive integer k, determines a path that has length at most L and has less than k edges, if such a path exists. We now define such an algorithm.

We construct a layered directed graph $D' = (V', A')$ in which the set V' consists of k copies of each vertex in V. We denote the ith copy, $1 \leq i \leq k$ of a vertex v in V by v_i. For every arc (u, v) in E, we have the following $k - 1$ arcs in A', each of which has the same length as that of (u, v): (u_i, v_{i+1}) for $1 \leq i < k$. In addition, for each v in V, we have the following $k - 1$ arcs in A', each of length zero: (u_i, u_{i+1}) for $1 \leq i < k$. It is now easy to see that there is a path in D from s to t that has length at most L and has at most k arcs if and only if there is a path in D' from s_1 to t_k that has length at most L. Moreover, given a path P' in D' from s_1 to t_k that has length at most L, we can construct, in polynomial time, a path in D from s to t that has length at most L and has at most k arcs. To complete the description of the algorithm, we note that we can determine the existence of path P' by finding the shortest path between s_1 and t_k.

In [16], [20], and [22], the greedy algorithm is analyzed to show that it satisfies at least $O_u/\sqrt{m}$ demands for an instance of EDP (also see [5]). (Recall that O_u denotes the optimum value for a given USF instance.) Their arguments may be extended easily to give $\sqrt{u(A)}$-approximations for unsplittable problems with general demands and capacitated networks (where $u(A) = \sum_e u(e)$). We improve this bound for capacitated networks by invoking the greedy algorithm several times.

Theorem 10 *Consider a unit-profit instance of* USF *in a directed graph D. If $d_{max} \leq d_{min}\Delta$ for some integer Δ, then the greedy algorithm may be used to find a solution which satisfies at least $\frac{O_u}{2\Delta\sqrt{m}}$ demands.*

Proof: Consider a guess O for O_u and set $u_e^O = \min\{u_e, \frac{d_{min}O}{\sqrt{m}}\}$. Call an arc *clipped* if $u_e^O \neq u_e$. We associate a length l_e with each arc in the following manner: e is assigned a length of 0 if it is clipped and 1 otherwise. Run the greedy algorithm on the graph D^O obtained by reducing the capacity of each arc e to u_e^O. Let $P_1, P_2, \ldots, P_t$ be the flow paths chosen by the greedy algorithm in D^O.

Let $\mathcal{Q}$ be those demands which are satisfied in some optimal solution but not by the greedy algorithm. Let Q be a path in the optimal solution satisfying some demand in $\mathcal{Q}$. It follows that there is an arc e in D such that (i) Q uses the arc e, and (ii) the greedy

algorithm is using at least a capacity of $u_e^O - d_{max}$ on the arc e. Otherwise, the greedy algorithm could have satisfied this demand along the flow path Q. Now if e were a clipped arc then the greedy algorithm must have routed $(\frac{Od_{min}}{\sqrt{m}} - d_{max})/d_{max} \geq \frac{O}{2\Delta\sqrt{m}}$ demands and we are done. So we may assume this is not the case and hence each such Q must contain an arc of length 1 from D^O that is in common with some P_i.

We now build on the ideas used in [20, 22] and analyze how many demands in the set $\mathcal{Q}$ are not satisfied due to a greedily selected flow path P_i. Let n_i denote the number of arcs of length 1 in the greedy flow path P_i. Consider any arc e and the greedy flow paths routed through it. We can imagine e as a bin of capacity u_e^O such that each successive greedy flow path is allocated a contiguous block of capacity on e. In an analogous manner, the optimal flow paths corresponding to the demands in Q can be viewed to have been allocated successive contiguous blocks of capacity on e. A greedy flow path P_i is said to *block* an optimal flow path Q if i is the least index such that (a) P_i and Q share a common arc e of length 1, and (b) the capacity block of P_i on e overlaps with the capacity block of path Q. Since each blocked optimal path Q uses up a capacity of d_{min} at least and each greedy path P_i uses a capacity of d_{max} at most, we deduce that any P_i may block at most $n_i\lceil d_{max}/d_{min}\rceil \leq n_i\Delta$ flow paths in $\mathcal{Q}$. Let k_i denote the number of optimal flow paths (corresponding to demands in $\mathcal{Q}$) that are blocked by P_i. It follows that $k_i \leq n_i\Delta$. But also by the definition of the greedy algorithm, we have that each such blocked flow must have used at least $n_i d_{min} \geq (k_i/\Delta)d_{min}$ units of length 1 capacity. Hence the total length 1 capacity used by the unrouted paths from the optimal solution is at least $\frac{d_{min}}{\Delta}\sum_i k_i^2$. Combining this with the observations that all length 1 arcs are unclipped and that the total available length 1 capacity is at most $\sum_{e\in A(D)} u_e^O$, we obtain

$$\frac{d_{min}}{\Delta} \cdot \frac{(\sum_{i=1}^{t} k_i)^2}{t} \leq \frac{d_{min}}{\Delta}\sum_{i=1}^{t} k_i^2 \leq \sum_{e\in A(D)} u_e^O \leq \sqrt{m}Od_{min}. \tag{6}$$

Since $\sum_i k_i = |\mathcal{Q}| = O_u - t$, we obtain $\frac{(O_u-t)^2}{t} \leq \Delta\sqrt{m}O$. Running the above procedure for various possible values of O (guessed within a factor of 2 by doubling), we may deduce that $t \geq \frac{O_u}{2\Delta\sqrt{m}}$ as required. ∎

If $\Delta > 2$, we can divide the demands into $O(\log\Delta)$ classes and do the above analysis for the class that has the largest number of pairs satisfied by an optimal solution. This would give us a solution that satisfies at least $O(\frac{O_u}{\log\Delta\sqrt{m}})$ demands. The preceding argument applies equally well (with $\Delta = 1$ of course) to BLEDP and so we obtain the following result.

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Theorem 11 *There is a polynomial-time $O(\sqrt{m})$ approximation algorithm for* BLEDP. ■

At first glance, it may appear that the proof of Theorem 10 also gives a similar bound for the integral splittable problem. There is a snag, however; we may only deduce the following.

Theorem 12 *Consider a unit-profit instance of the* ISF *problem. Then by applying the greedy algorithm $O(\log n)$ times, we may find a solution which satisfies at least $\frac{O_{is}}{2\sqrt{md_{max}}}$ demands.*

Proof: Mimic the proof of Theorem 10 directly, except clip each capacity at $\frac{O\sqrt{d_{max}}}{\sqrt{m}}$. The proof breaks down only because we cannot guarantee that the flow P_i blocks at most $n_i\Delta$ flow systems from an integral splittable optimum. Indeed, we can only deduce that P_i must use at least k_i units of length 1 capacity, and hence so do each of the flows which it blocks. Equation (6) becomes $\sum_i k_i^2 \leq \sqrt{md_{max}}O$, and the rest then follows. ■

Returning to the general versions with profits and arbitrary demands, the repeated greedy algorithm, together with techniques from the proof of Theorem 6, gives us an upper bound on the approximation factor that nearly matches the hardness result of Corollary 3.1.

Corollary 5.1 *The general* USF *problem with arbitrary polynomially bounded demands can be approximated to within a factor of $O(\sqrt{m}\log^2 m)$. The general* ISF *problem with arbitrary polynomially bounded demands can be approximated to within a factor of $O(\sqrt{md_{max}}\log^2 m)$.* ■

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