Formulations for the Stable Set Polytope

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Abstract. We give a simple algorithm for the weighted stable set problem of an arbitrary graph which yields an extended formulation for its stable set polytope. The algorithm runs in polynomial time for the class of *distance claw-free graphs.* These are the graphs such that for each node, neither its neighbour set nor the set of nodes at distance two contain a stable set of size three. The extended formulation we obtain is of polynomial size for distance claw-free graphs. These graphs are of interest from the point of view of their stable set polyhedra due to the fact that all of the complicated necessary inequalities given by Giles and Trotter for the polyhedra of claw-free graphs are also necessary for the class of distance claw-free graphs.

Keywords: *Stable set polytope, compact formulation, projection, claw-free graph.*

1 Introduction

Given a graph G and a vector $w \in \mathbf{Q}^V$ of node weights, the *weighted stable set problem* is to find, a stable set (set of mutually nonadjacent nodes) of maximum weight. The *stable set polytope* of G, denoted by $P(G)$, is the convex hull of the incidence vectors of its stable sets, i.e., $P(G) = conv(\{\chi^S : S \text{ is a stable set of } G\})$. A linear system $Ax \leq b$, is *defining* for $P(G)$ if $P(G) = \{x : Ax \leq b\}$ and hence the original optimization problem is equivalent to the linear program $\max\{w \cdot x : Ax \leq b\}$. In particular, such a system immediately yields an optimality criterion through linear programming duality.

The stable set problem is NP-hard. Therefore one would not expect to find a 'good' defining linear system for the stable set polytopes of general graphs. Here 'good' means one for which the validity of each proposed inequality can be verified in polynomial time. If such a system existed, this would establish that $NP = co - NP$, which most researchers view as highly unlikely. There are certain classes of graphs, however, for which such systems are known, for example, bipartite graphs, line graphs [9], series-parallel graphs [5] and graphs with no stable set of size three (Cook [8], see [17]). The weighted stable set problem for each of these classes is polynomially solvable. Indeed the development of a weighted algorithm has traditionally occurred in conjunction with or led to a defining linear system for the corresponding polytope.

The most notable exception to this at present is the class of *claw-free* graphs. These are graphs for which no node's neighbourhood contains a stable set of size three (i.e., $\alpha_{N(v)} \leq 2$

for each node v). This is equivalent to requiring that a graph have no induced subgraph isomorphic to the graph in Figure 1. In 1977, Minty [15] gave a polynomial time algorithm to solve the weighted stable set problem for the class of claw-free graphs. Note that any line graph is claw-free and so the stable set problem for claw-free graphs generalizes the matching problem. Hence Minty's algorithm can be viewed as an extension of the classical blossom algorithm of Edmonds. (The *line graph* of G, denoted by $L(G)$, is the graph whose nodes are the edges of G; two nodes of $L(G)$ are adjacent if the corresponding edges of G are incident. It follows that there is a one-one mapping between the matchings of G and the stable sets of $L(G)$.) Minty also showed that if the bound on $\alpha_{N(v)}$ is changed from 2 to 3, then the stable set problem for the resulting class of graphs is NP-hard.

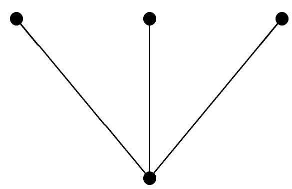

Figure 1: A claw

The strongest positive results towards finding defining systems for claw-free graphs appear in the thesis of Ben Rebea [4]. There, a linear description is proposed for a subclass of the claw-free graphs: the *quasi-line graphs*. This class consists of the graphs for which the neighbourhood of each node can be partitioned into two cliques. Note that this class properly contains the class of line graphs. Tragically, Ben Rebea was killed in an automobile accident shortly after completing his thesis. Since then, several people have tried to prepare the results in his thesis for publication, however, none of these attempts has been successful.

Giles and Trotter [10] showed that defining systems for stable set polytopes of claw-free graphs must be substantially more complex than those for matching polytopes. They showed that any (complete) defining system for claw-free stable set polyhedra must include several classes of complicated inequalities. Moreover, when the coefficients in these inequalities were scaled to be relatively prime integers, arbitrarily large coefficients could be present. These inequalities provided counterexamples to several conjectures existing at the time.

In the current paper we discuss the class of *distance claw-free* graphs [16]. These are claw-free graphs such that for each node v, the nodes at distance exactly two from v do not contain a stable set of size three. It is straightforward to verify that this class is incomparable with the class of quasi-line graphs. Although the distance claw-free graphs have a much simpler structure than claw-free graphs, all of the complicated facet-inducing inequalities for claw-free graphs appearing in the literature are also necessary for the distance claw-free graphs.

In Section 4 we give a conceptually simple algorithm, based on dynamic programming, to solve the weighted stable set problem for a general graph. We show that the algorithm has polynomially bounded running time if the input is restricted to the class of distance claw-free graphs. This together with the fact that the complications illuminated in [10] remain valid for distance claw-free graphs, suggests that this class can be a good starting point for finding claw-free defining systems.

The above algorithm also yields a description of a polyhedron which contains $P(G)$ as a projection. In the case of distance claw-free graphs, this gives a compact formulation for $P(G)$. (A formulation is *compact* if the number of variables and constraints both grow polynomially with the size of the graph.) This is of interest as such formulations are not known for stable set polyhedra in general and in fact not even for the special case of matching polyhedra. Moreover, such formulations make possible the application of projection techniques as developed by Balas and Pulleyblank [1], in order to obtain a good linear description for stable set polyhedra of the distance claw-free graphs.

2 Distance Claw-Free Graphs

Recall that a graph is distance claw-free if for each node there is no stable set of size three contained entirely in its neighbourhood $N(v)$ or in its second neighbourhood $N_2(v)$ (those nodes at distance **exactly** two). In this section we describe some of the facet-inducing inequalities given by Giles and Trotter [10] and show that the graphs from which they arise lie in the more restricted class of distance claw-free graphs.

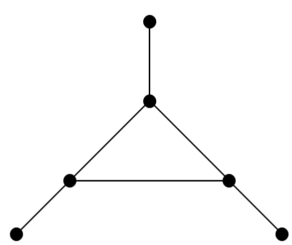

Figure 2: A net

To this end, call a graph *claw-free, net-free* or CN-free if it is claw-free and contains no induced subgraph isomorphic to that of Figure 2 - called a *net*. In [16], a structural characterization is given for CN-free graphs. For our present purposes we need only the following result.

Theorem 2.1 *If G is CN-free, then it is distance claw-free.*

Hammer et al. [14] give a polynomial time algorithm to solve the stable set problem for the class of CN-free graphs. We later describe a polynomial time algorithm to solve the weighted stable set problem for distance claw-free graphs which relies on the following property for those graphs.

Theorem 2.2 *If G is distance claw-free, then for each node v and integer i, $\alpha_{N_i(v)} \leq 2$.*

Proof: Suppose that G is distance claw-free and that v is a node such that $N_i(v)$ contains a stable set of size three x_1, x_2, x_3. Choose v and i so that i is minimized. By definition of a distance claw-free graph $i > 2$ and so no pair of the x_j's has a common neighbour y in $N_{i-1}(v)$ since y would have a neighbour in $N_{i-2}(v)$ and so would center a claw. Thus we may choose distinct y_1, y_2, y_3 in $N_{i-1}(v)$ such that $N(y_j) \cap \{x_1, x_2, x_3\} = \{x_i\}$ for each j. Now by minimality of i we may assume that at least the edge y_1y_2 exists amongst the y_j's and we let z be any neighbour of y_1 in $N_{i-2}(v)$. Since y_1 does not center a claw, we must also have $zy_2 \in E(G)$. Note next that $y_3z \notin E(G)$ however, since then $N_2(z)$ would contain the stable set $\{x_1, x_2, x_3\}$. This then implies that $y_3y_j \notin E(G)$ for $j = 1, 2$ for otherwise y_j would be adjacent to the stable set y_3, z, x_j. Thus the shortest path from z to y_3 in the graph induced by $\{y_3\} \cup (\cup_{k=0}^{k=i-2} N_k(v))$ is length at least two and so there is a node $w \in N_2(z)$ which lies in some $N_k(v)$ for $k \leq i-2$. This implies that the second neighbourhood of z contains the stable set w, x_1, x_2, a contradiction. □

3 Complicated Facets

Let $a \cdot x \leq \gamma$ be a facet defining inequality for $P(G)$. Since $P(G)$ is full dimensional, another inequality induces the same facet if and only of it is a positive multiple of $a \cdot x \leq \gamma$. Since all vertices of $P(G)$ are rational (in fact, integral) every facet is induced by an inequality with integral coordinates. We say that an inequality is in *standard form* if all coefficients and the righthand side are relatively prime integers. An inequality is a *rank inequality* if all coefficients are 1, when the inequality is in standard form.

We now use the Giles-Trotter inequalities to show:

(1) There exist infinitely many distance claw-free graphs G such that $G_{N(v)}$ is perfect for each node v, yet for which $P(G)$ has a nontrivial, nonrank facet. ($G_{N(v)}$ is the subgraph of G induced by $N(v)$.)

(2) For each integer k, there exist a distance claw-free graph G, and an inequality $a \cdot x \leq \gamma$, facet inducing for $P(G)$, in standard form such that $a, \gamma \geq k$.

(3) There are infinitely many distance claw-free graphs G and $a \in \mathbf{Q}^V$ such that $a \cdot x \leq \bar{1}$ is facet-inducing for $P(G)$ and such that for all $X \subseteq V$, $a[X] \cdot x \leq \bar{1}$ does not induce a rank facet of $P(G_X)$.

(4) For each integer k there is a distance claw-free graph G such that the Chvátal rank (defined later) of $P(G)$ with respect to the fractional stable set polytope of G is greater than k.

Recall that a circulant C_n^k is a graph obtained from an n-cycle by letting each node of the cycle be adjacent to the k closest nodes in either direction on the cycle, i.e., $V(C_n^k) =$

$\{0, \ldots, n-1\}$ and for $i \neq j$, $u_i u_j \in E(C_n^{k+1})$ if $(|i-j| mod\ n) \leq (k+1)$. For $k = 1, 2 \ldots$ we construct a graph $G^k = (V^k, E^k)$. Set $n = 2k(k+2)+1$ Consider the two graphs $H^1 \equiv C_n^{k+1}$ and $H^2 \equiv C_n^k$ with node sets $\{u_0, \ldots u_{n-1}\}$ and $\{v_0, \ldots v_{n-1}\}$ respectively. We have that $V^k = \{u_i, v_i\}_{i=0}^{n-1}$ and that $E^k = E(H^1) \cup E(H^2) \cup \{u_i v_j : ((j-i)\ mod\ n) \leq (2k+1)\}$. Note that we do not take the absolute value of $(j-i)$ and so u_i is adjacent to $v_i, v_{i+1}, \ldots$ etc. (see Figure 3 for G^1).

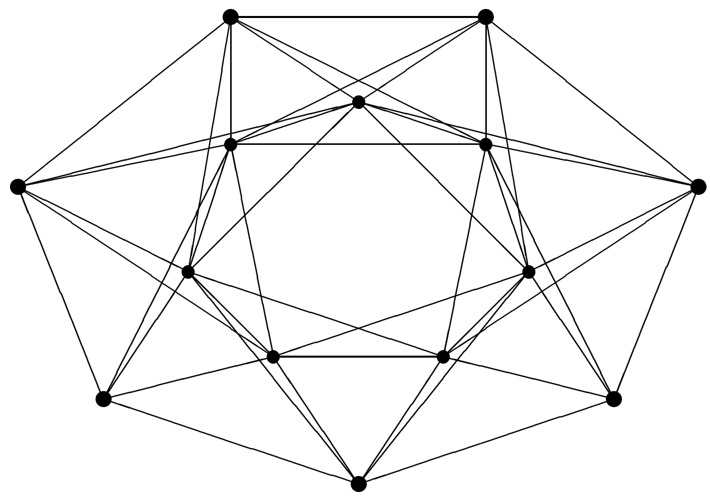

Figure 3: Graph G^1

Giles and Trotter prove the following.

Theorem 3.1 (Giles, Trotter [10]) *For $k = 1, 2 \ldots$,*

$$(k+1)\sum_{i=0}^{n-1} x_{u_i} + k \sum_{i=0}^{n-1} x_{v_i} \leq 2k(k+1) \tag{5}$$

induces a facet of $P(G^k)$.

Giles and Trotter also show that for each node $v \in V^k$, the neighbours of v partition into two cliques. This shows that the neighbours of v induce a perfect graph and also that G^k is claw-free.

It is straightforward (but tedious) to show that no G^k contains an induced net, thus the facet-inducing inequalities described in Theorem 3.1 are necessary even when restricting ourselves to distance claw-free graphs. Thus the family $G^1, G^2, \ldots$ verifies (1) and (2).

Giles and Trotter describe how to construct infinitely many claw-free graphs (called *wedges*) with the property described in (3). We show only one of these examples. Let G be a graph whose *complement* is the graph of Figure 4. Let $a \cdot x \leq 3$ be the inequality given by:

$$\sum_{i=1}^{5} x_i + 2\sum_{i=6}^{8} x_i \leq 3.$$

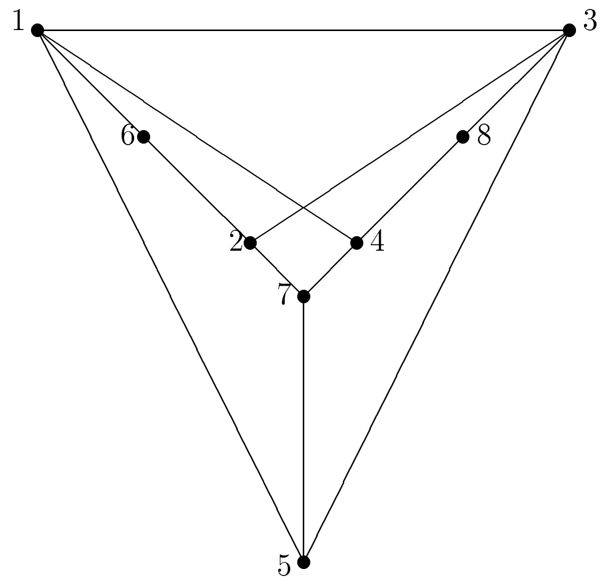

Figure 4:

It is easy to verify that $a \cdot x \leq 3$ is facet-inducing for $P(G)$. Furthermore, no projection $a[X] \cdot x \leq 3$ is a rank facet for $P(G_X)$ since G_X would have to have rank 3. This is impossible since G has only one stable set of size three, i.e., the above graph has only one triangle. This also implies that G is is CN-free since every node of Figure 4 is adjacent to the unique triangle and since the complement of a net contains four triangles. Hence G is also distance claw-free by Theorem 2.1. Note, however, that G is not a quasi-line graph since the node 8 contains a 5-cycle $1, 7, 6, 5, 2$ in its neighbourhood.

We describe briefly the rounding procedure for generating valid inequalities for an integral polyhedron. Let $P = \{x : Ax \leq b\}$ be a polyhedron in $\mathbf{Q}^V$. Let P_I denote the integer hull, i.e., the convex hull of the integer valued members of P. It is clear that for $y \geq 0$ such that $yA \in \mathbf{Z}^V$, we have that

$$(yA)x \leq \lfloor y \cdot b \rfloor$$

is a valid inequality for P_I. Such an inequality is said to be generated from P by *round-down.* Gomory [11],[12],[13] introduced such inequalities for his algorithm to solve integer programming problems; Chvátal [6], developed the theory behind these inequalities. Some of the round-down inequalities may induce facets of P_I. Let $P^{(1)}$ be the members of P which satisfy all of the round-down inequalities. For $k \geq 1$ let $P^{(k+1)} = (P^{(k)})^{(1)}$. Then we have

$$P \supseteq P^{(1)} \supseteq P^{(2)} \ldots \supseteq P_I.$$

Theorem 3.2 (Chvátal [6]) *For any polyhedron P, there is some integer t such that $P^{(t)} = P_I$.*

The smallest integer t such that $P^{(t)} = P_I$ is called the *Chvátal rank* of P.

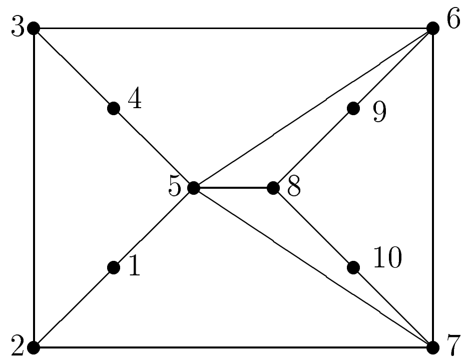

Figure 5:

For the purpose of studying the stable set polytope of a graph G, one often begins with the *fractional stable set polytope* $P = \{x : x \geq 0,\ x(K) \leq 1 \quad \text{for each clique } K\}$. It is clear that $P_I = P(G)$. Giles and Trotter show that there are claw-free graphs for which P has rank at least two. For example, consider the graph G whose complement is given in Figure 5. (Note again that G is CN-free but not a quasi-line graph.) One checks that $a \cdot x \equiv 2\sum_{i=1}^{5} x_i + x_6 + x_7 + x_8 + 3(x_9 + x_{10}) \leq 4$ is facet-inducing for $P(G)$. Yet $a \cdot x \leq 4$ cannot be generated by round-down from the clique inequalities. To see this, let A be the clique-node incidence matrix of G and set $x^0 = \frac{1}{2}\chi^{\{1,2,3,4,5\}}$. If $yA = a$, then

$$5 = a \cdot x^0 = (yA)x^0 \leq y \cdot \bar{1},$$

and so $\lfloor y \cdot \bar{1} \rfloor \geq 5$.

Chvátal [7] gives a strengthening of this negative result in that he has shown that for each integer k there is a graph G with $\alpha = 2$ such that the Chvátal rank of the fractional stable set polytope of G is greater than k. This clearly implies (4).

We have seen that although the class of distance claw-free graphs is a proper subclass of the class of claw-free graphs, there is evidence that describing the stable set polytope may already be a difficult task. In particular, we have shown that many existing examples of complicated facets for stable set polytopes of claw-free graphs remain facet inducing for the stable set polytopes of distance claw-free graphs, and in fact also for the class of CN-free graphs. This suggests that these smaller classes may be a good point to embark on the search for defining linear systems. Moreover, these smaller classes possess several simplifying structural properties. For instance, any CN-free graph G satisfies the following condition: if H is an odd hole of G, then $V = H \cup N(H)$.

4 Dynamic Programming and the Stable Set Problem

We give a dynamic programming algorithm which solves the weighted stable set problem for any graph. In general, the algorithm will not have polynomially bounded running time.

However, the algorithm will terminate in polynomial time for a class of graphs which includes the distance claw-free graphs.

We begin with a reduction of the stable set problem for an arbitrary graph G and weights $w \in \mathbf{Q}^V$, to a longest path problem in a derived directed graph. For node z and for $i = 0, 1 \ldots,$ let $N_i(z)$ denote the set of nodes which are at distance i, in G, from z. We fix a node z and let ρ be the largest integer i such that $N_i(z) \neq \phi$. For any graph G, we let $\mathcal{S}(G)$ denote the set of all stable sets of G. We denote by $\mathcal{S}_i$, the set $\mathcal{S}(G_{N_i(z)})$. We create a directed graph $D_z(G)$ or simply $D(G)$ as follows. The nodes of $D(G)$ consist of $\{v_S^i : S \in \mathcal{S}_i \text{ for some } i = 0, \ldots \rho\}$ together with special nodes u^*, v^*. The arcs of $D(G)$ are given by (see Figure 6):

- for each $S \in \mathcal{S}_\rho$, $(v_S^\rho, v^*) \in A$,
- the arcs (u^*, v_ϕ^0) and $(u^*, v_{\{z\}}^0)$,
- for $i = 0, \ldots \rho - 1$ and each stable set S contained in $N_i(z) \cup N_{i+1}(z)$, $(v^i_{S \cap N_i(z)}, v^{i+1}_{S \cap N_{i+1}(z)}) \in A$.

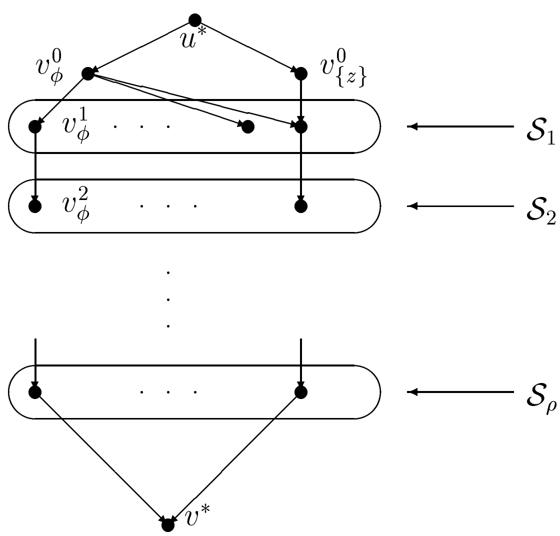

Figure 6: $D(G)$

We assign weights w' to the arcs of $D(G)$ as follows:

- for each arc $\beta = (x, v^*)$, $w'_\beta = 0$,
- for each arc $\beta = (x, v^i_{S'})$, $w'_\beta = \sum_{y \in S'} w_y$.

It is clear that any maximal directed path P, is of the form $u^*, v^0_{S^0}, v^1_{S^1}, \ldots v^\rho_{S^\rho}, v^*$ where $\cup_{i=0}^\rho S^i$ is a stable set of G. This stable set is denoted by $\pi(P)$. In fact π is a one-one mapping. That is for any stable set S of G, there is a unique maximal length directed path of $D(G)$, denoted by $\pi^{-1}(S)$. Hence, $\pi(\pi^{-1}(S)) = S$ and $w(S) = w'(\pi^{-1}(S))$. Thus finding a maximum weight stable set in G is equivalent to finding a longest directed (u^*, v^*)-path in $D(G)$. Since this graph is acyclic, this latter problem can be solved using the polynomial time algorithm offered by Bellman [3]. Of course the number of iterations that this algorithm requires depends on the number of nodes in $D(G)$. If $\mathcal{G}$ is a class of graphs for which there is a constant c such that for each $G \in \mathcal{G}$ there is some node $z \in V$ with $\alpha_{N_i(z)} \leq c$ for each i, then the above algorithm will terminate in at most $|V|^{c+1}$ iterations for any graph $G \in \mathcal{G}$. Theorem 2.2 now implies that the size of $D(G)$ is polynomially bounded in the size of G for distance claw-free graphs. Hence the dynamic programming approach yields a polynomially bounded algorithm for the weighted stable set problem for distance claw-free graphs.

5 The Extended Formulation

5.1 The Lifted Polyhedron

We carry the analysis of this dynamic programming approach further to give a linear defining system of a higher dimensional polyhedron (i.e., one which contains more variables than the number of nodes in our original graph) with the property that $P(G)$ is obtained by projecting onto the variables corresponding to V. A defining linear system such as this is called an *extended formulation* of $P(G)$. It is a well known open problem to determine whether the matching polyhedron has an extended formulation with a number of variables polynomially bounded in the size of the graph. Such a system is called a *compact formulation* for $P(G)$.

Let B be the arc-node incidence matrix of the digraph D, i.e., for each arc $\beta = (u, v)$, $B_{(\beta,v)} = 1$, $B_{(\beta,u)} = -1$ and for each other node y, $B_{(\beta,y)} = 0$. It is then straightforward to see that

$$(6) \qquad \begin{aligned} \min\{&x_{v^*}\} \\ B \cdot x \geq\ & w' \\ x \geq\ & 0 \end{aligned}$$

gives exactly the value of a maximum weight stable set in G (relative to w). It is well known that if w is integral, then (6) has an integral optimal solution. (This is because B is totally unimodular). It is also true that the dual

$$(7) \qquad \begin{aligned} \max\{&w' \cdot \ \mu\} \\ \mu \cdot B \leq\ & \chi^{\{v^*\}} \\ \mu \geq\ & 0 \end{aligned}$$

has an integral optimal solution. Note that any integral solution μ, to (7), must satisfy $\mu \cdot B = \chi^{\{v^*\}} - \chi^{\{u^*\}}$. It follows that such a solution is the incidence vector of a directed (u^*, v^*)-path P, in D. Hence $\pi(P)$ is a maximum w-weight stable set of G. We can obtain the incidence vector of this stable set through the following equations:

$$\chi_v^{\pi(P)} = \sum_{(u,v_S^i):v\in S} \mu_{(u,v_S^i)}.$$

Let C be the $V_G \times A$ matrix such that $C_{(v,\beta)} = 1$ if β is of the form (u, v_S^i) such that $v \in S$; otherwise $C_{(v,\beta)} = 0$. Hence we have that $\chi^{\pi(P)} = C \cdot \mu$.

We define the following polyhedron in $\mathbf{Q}^{V\cup A}$:

$$H \equiv \left\{ (y,\mu) : \begin{array}{c} \mu \geq 0 \\ \mu \cdot B \leq \chi^{\{v^*\}} \\ y - C \cdot \mu = 0 \end{array} \right\}. \tag{8}$$

The polytope $P(G)$ is a projection of H, for

$$P(G) = \{y \in \mathbf{Q}^V : \text{there exists } \mu \in \mathbf{Q}^A \text{ such that } (y,\mu) \in H\}. \tag{9}$$

It is easy to see that $P(G)$ is contained in the right hand side of (9). This is because for any stable set S, $(\chi^S, \chi^{\pi^{-1}(S)}) \in H$. Conversely, suppose that $(y,\mu) \in H$. Since B is totally unimodular it follows that H is integral. Hence (y,μ) is a convex combination of integral vectors $(y_1,\mu_1),\ldots(y_k,\mu_k)$, say, in H. But as we have seen each of the y_i's must be the incidence vector of a stable set of G. Hence $y \in P(G)$. We refer to H as the *lifted polyhedron.*

A system such as (8) which defines a lifted polyhedron of $P(G)$ is called an *extended formulation* of $P(G)$. It is a well known open problem to determine whether the matching polyhedron has an extended formulation with a polynomial number of variables. Such a system is called a *compact formulation.* The following is a consequence of Theorem 2.2.

Theorem 5.1 *The system given in (8) is a compact formulation for the stable set polytope of distance claw-free graphs.*

The conclusion of the above theorem need not hold for the class of claw-free graphs (or even the quasi-line-graphs). (In fact, a compact formulation for the claw-free graphs would immediately yield such a system for the matching polyhedron.) This is because for general claw-free graphs, there is no constant bound for $\alpha_{N_2(v)}$. It is true, however, that for each k, there is a compact formulation for the stable set polytopes of graphs which contain a node v such that $\alpha_{N_i(v)} \leq k$ for each integer i.

An extended formulation is called *symmetric* if for each permutation of the nodes (variables for the projected polyhedron) there is a permutation of the lifted variables so that the lifted polyhedron remains invariant. Yannakakis shows the following:

Theorem 5.2 (Yannakakis [18]) *There is no symmetric compact formulation of the perfect matching polytope.*

(Here, the perfect matching polytope is the convex hull of incidence vectors of perfect matchings in the complete graph.)

5.2 On Extreme Rays

We now show how the facet-inducing inequalities for $P(G)$ arise from a certain cone associated with H. Consider the cone $\{(\zeta,\eta) : \zeta \cdot B^T - \eta \cdot C \geq 0,\ \zeta \geq 0\}$. Now suppose that $\hat{W}$ is any set of generators of this cone. The following is then immediate from results of Balas and Pulleyblank.

Theorem 5.3 (Balas,Pulleyblank [1]) $P(G) = \{x \in \mathbf{Q}^V : \eta \cdot x \leq \zeta_{v^*}$ *for all* $(\zeta, \eta) \in \hat{W}\}$.

Theorem 5.3 shows that a description of the generators $\hat{W}$ would yield a description of the facets of $P(G)$. In addition, any nontrivial facet for $P(G)$ has nonnegative coefficients and so we need only consider $(\zeta, \eta) \in \hat{W}$ for which $\eta \geq 0$. Thus we let W be the cone

$$\{(\zeta, \eta) : \zeta \cdot B^T - \eta \cdot C \geq 0, \quad \zeta, \eta \geq 0\}. \tag{10}$$

Henceforth, we discuss how to determine a set of generators of W which is sufficient to derive all of the facet-inducing inequalities for $P(G)$.

We denote by $I^=(\zeta, \eta)$ the set of inequalities in (10) which are satisfied with equality by (ζ, η). Let $\mathcal{R}$ be the set of maximal proper subsets R such that $R = I^=(\zeta, \eta)$ for some $(\zeta, \eta) \in W$. For $R \in \mathcal{R}$, let $\mathcal{T}(R)$ denote the members (ζ, η) of W for which $I^=(\zeta, \eta) = R$. Since W is pointed it follows that each set $\mathcal{T}(R)$ consists of the positive multiples of some member of W. These sets are called the *extreme rays* of the cone W. (We will also refer to a representative of one of these sets as an extreme ray.) It is well known that $\hat{W}$ generates W if and only if for each $R \in \mathcal{R}$, some member of $\mathcal{T}(R)$ is contained in $\hat{W}$.

For a node $u \in D$, the *arc inequalities for* u are the inequalities from (10) associated with the arcs whose heads are the node u. We call an arc *tight* for $(\zeta, \eta) \in W$ if its corresponding constraint is tight. The following proposition is used to give sufficient conditions for a vector in W to be extreme.

Proposition 5.4 *If* $(\hat{\zeta}, \hat{\eta}) \in W$, $\zeta_{u^*} = 0$, *and for each* $u \in V_D - \{u^*\}$ *one of* u*'s arc inequalities is tight, then the set of tight arcs for* $(\hat{\zeta}, \hat{\eta})$ *is maximal over all* $(\zeta, \hat{\eta}) \in W$.

Proof: Suppose that the set of tight arcs for some $(\zeta, \hat{\eta}) \in W$ properly contains the corresponding set of arcs for $(\hat{\zeta}, \hat{\eta})$. Note that this implies that $\zeta_{u^*} = 0$. We let $C(\hat{\eta})$ be the set of nodes v^i_S such that $\hat{\eta}(S) > 0$. For each such node there is some u such that $(u, v^i_S) \in A$. Thus we must have

$$\zeta_{v^i_S} \geq \zeta_u + \hat{\eta}(S) > 0. \tag{11}$$

Since $\hat{\eta} \geq 0$, the node weights ζ, along any directed path must be nondecreasing (by examination of the arc inequalities (11)). Let $C^+(\hat{\eta})$ be those nodes v in D for which there exists a node $u \in C(\hat{\eta})$ and a directed (u, v)-path, i.e., the descendants of $C(\hat{\eta})$. It follows from (11) that $\zeta_u > 0$ for each $u \in C^+(\hat{\eta})$. But $\hat{\zeta}_u = 0$ for each $u \notin C^+(\hat{\eta})$. Thus the set of tight arcs for $(\zeta, \hat{\eta})$ must properly contain the tight arcs for $(\hat{\zeta}, \hat{\eta})$. Note that for any node $v^k_{\mathcal{S}_k} \in D$, there is a directed $(u^*, v^k_{\mathcal{S}_k})$-path, $u^*, v^1_{\mathcal{S}_1}, v^2_{\mathcal{S}_2}, \ldots v^k_{\mathcal{S}_k}$ consisting only of tight arcs for $(\hat{\zeta}, \hat{\eta})$. Note that these arcs are also tight for $(\zeta, \hat{\eta})$. It follows that

$$\zeta_{v^k_{\mathcal{S}_k}} = \sum_{i=1}^{k} \hat{\eta}(\mathcal{S}_i). \tag{12}$$

Thus $\hat{\zeta} = \zeta$ and the result follows. □

We have the following immediate corollary.

Corollary 5.5 *Suppose that $(\hat{\zeta}, \hat{\eta}) \in W$, $\zeta_{u^*} = 0$, and each $u \in V_D - \{u^*\}$ has some tight arc inequality, then if $\hat{\eta} \cdot x \leq \hat{\zeta}_{v^*}$ is facet-inducing for $P(G)$, then $(\hat{\zeta}, \hat{\eta})$ is an extreme ray of W.*

We now describe a procedure which, given a facet-inducing inequality $\hat{\eta} \cdot x \leq \hat{\zeta}_{v^*}$ for $P(G)$, will produce the other values of $\hat{\zeta}$ so that $(\hat{\zeta}, \hat{\eta})$ is an extreme ray of W.

> Algorithm: **MAKE.RAY**
> Set $\hat{\zeta}_{u^*} = 0$
> For $i = 0, \ldots \rho$
>   For each $S \in \mathcal{S}_i$
>   Set $\hat{\zeta}_{v^i_S} = \max\{\hat{\zeta}_u : (u, v^i_S) \in A\} + \hat{\eta}(S)$
>   End
> End

Note that for each node u, **MAKE.RAY** chooses $\hat{\zeta}_u$ so that each of u's arc inequalities is satisfied by $(\hat{\zeta}, \hat{\eta})$. Furthermore, each node other than u^* has some tight arc inequality. Since $\hat{\eta} \cdot x \leq \hat{\zeta}_{v^*}$ is facet-inducing, there is at least one extreme ray $(\zeta, \hat{\eta})$ in W. Hence by Corollary 5.5, $(\hat{\zeta}, \hat{\eta})$ must be extreme.

5.3 Extendable Vectors

We call $\zeta \in \mathbf{Z}^{V_D}$ *extendable* if there is some $\eta \in \mathbf{Z}^{V_G}$ such that $(\zeta, \eta) \in W$, $gcd(\{\zeta_u\}_{u \in V_D}) = 1$ and for each $u \in V_D$, at least one of u's arc inequalities is tight for (ζ, η). We say that ζ *extends to* η or to the inequality $\eta \cdot x \leq \zeta_{v^*}$. Note that (12) implies that $gcd(\{\hat{\zeta}_u\}_{u \in V_D}) = 1$ if and only if $gcd(\{\hat{\eta}_v\}_{v \in V_D}) = 1$. Therefore, our analysis of **MAKE.RAY** shows that

(13) There is a one-one correspondence between the nontrivial facets of $P(G)$ and the set of extendable vectors.

Thus one can attempt to find a linear description for $P(G)$ by solving the following problem:

Problem *Characterize the set of extendable vectors.*

If this is to reduce the difficulty of our task, then it must be simpler to understand the structure of the derived graph D and in particular the extendable vectors. It is not enough that we can construct the extendable vectors given the (facet-inducing) inequalities. (For example, we could work backwards from the Giles-Trotter inequalities to find how they are derived from the extreme rays.) We must somehow understand the extendable vectors in and of themselves. An example of where this method has unraveled the complexity of the desired linear system is given in [2].

We start by examining the 0, 1 extendable vectors. For $C \subseteq V_D$, a *basis* for C is a maximal subset $B \subseteq C$ such that for each $u, v \in B$, there is no directed path from u to v or from v to u. In particular B is a stable set of D. Note that for any basis there is some integer i such that $B \subseteq (N_i(z) \cup N_{i+1}(z))$ (this is because one can construct a directed path through the nodes $v_\phi^1, v_\phi^2, \ldots$). The notion of a basis seems central to understanding the structure of extendable vectors. Indeed any extendable vector ζ, can be *decomposed* into its *characteristic bases* as follows: for $i = 1, \ldots$, let B_i be the basis of $\{v \in V_D : \zeta_v \geq i\}$. The *length* of such a decomposition is the number of nonempty bases B_i.

We start by displaying the following simple fact.

Proposition 5.6 *If ζ is extendable, then*

$$\zeta = \sum_{i \geq 1}^{l} \chi^{B_i^+},$$

where $B_1, B_2, \ldots B_l$ is a decomposition of ζ.

Proof: We note that if $\zeta_u \geq k$, then by maximality, $u \in B_1^+, \ldots B_k^+$. So suppose that $\zeta_u \leq k$ but that $u \in B_{k+1}^+$. This implies that there is some directed path from B_{k+1} to u whose ζ weights are **not** nondecreasing, contradicting the inequalities (10). □

Theorem 5.7 *A $0,1$ ζ (not equal to $\bar{1}$) is extendable if and only if there is some clique K such that $(\zeta, \chi^K) \in W$.*

Proof: First consider the clique inequality $x(K) \leq 1$, where K is a maximal clique of G. Let $B = \{v_S^i : |S \cap K| = 1\}$. Note that B is a stable set of D. We denote by B^+ the set of nodes in B which can be reached from some node of B (i.e., the descendants of B). It is easy to see that, given χ^K, **MAKE.RAY** sets $\hat{\zeta}_u = 1$ if and only $u \in B^+$. Hence χ^{B^+} extends to χ^K.

We now show that if ζ is $0,1$ valued (not equal to $\bar{1}$) and extends to some η, then $\eta = \chi^K$ for some clique K of G. Let $supp(\zeta) = \{v : \zeta_v = 1\}$. Let $B = \{v \in supp(\zeta) : \text{for all } (u, v) \in A, \zeta_u = 0\}$. It follows that

(14) $\qquad \eta_y = 1$ implies that $v_S^i \in B$ for any such node with $y \in S$.

The members of B correspond to stable sets $S_1, S_2, \ldots$. Since $(\zeta, \eta) \in W$, B is a basis. Thus $S_i \cup S_j$ is not a stable set for any S_i, S_j contained in different level sets. Hence if $y, y' \in V_G$ are in different level sets, then by (14) $\eta_y = 1 = \eta_{y'}$ implies that $yy' \in E$. Now suppose that y, y' are contained in one level $N_i(z)$. If $S = \{y, y'\}$ is a stable set, then $\eta(S) = 2$ and so every one of v_S^i's arc inequalities is violated by (ζ, η), a contradiction. Thus $\eta = \chi^K$ for some clique K as desired. □

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