Preselecting Homotopies for the Weighted Disjoint Paths Problem [1]

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Abstract: The work of Robertson and Seymour implies that the disjoint paths problem is polynomial solvable for a fixed number of terminals. Even, Itai and Shamir showed that a weighted version of the problem is NP-hard even for a demand graph consisting of two edges. In the present paper, it is shown that the weighted disjoint paths problem is polynomial solvable for graphs embedded on a fixed surface and fixed demand graph. An alternative formulation of the problem is: given a graph G embedded in a plane with a certain number of forbidden regions, find a maximum weight collection of pairwise internally disjoint paths connecting nodes as specified by some fixed demand graph. The results imply that this problem is polynomial solvable if the number of forbidden regions is bounded. One consequence is the existence of a polynomial-time algorithm for finding a maximum collection of induced paths between specified nodes in a graph embedded on a fixed surface.

Introduction

In this paper, $G = (V, E)$ denotes a graph without loops but possibly multiple edges and we set $n = |E(G)|$. For an excellent background on routing in graphs, the reader is referred to Frank [**?**]. It is well known that the work of Robertson and Seymour implies that for a fixed *demand graph* H the following problem is polynomially solvable:

Disjoint H-Paths (DP(H))

Instance: A graph G and a $1-1$ mapping $\phi : V(H) \to V(G)$.

Find: A family $\{P_e : e \in E(H)\}$ of internally node disjoint paths in G such that P_{uv} joins nodes $\phi(u)$ and $\phi(v)$.

The mapping Φ of course, merely fixes a set of *terminals* in G to act as the nodes of H. This paper addresses the weighted version of DP. For a demand vector $\mathbf{d} \in \mathbf{N}^{E(H)}$ of nonnegative integer weights on elements of $E(H)$, the $\mathbf{d}$-*multiple* of H, denoted by $H[\mathbf{d}]$, is the multigraph obtained by replacing each edge e by d_e parallel edges.

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Weighted Disjoint H-Paths (WDP(H))

INSTANCE: A graph G, demands $\mathbf{d} \in \mathbf{N}^{E(H)}$, and $1-1$ mapping $\phi : V(H) \to V(G)$.

FIND: A family $\{P_e : e \in E(H[\mathbf{d}])\}$ of internally node disjoint [2] paths in G such that P_{uv} joins nodes $\phi(u)$ and $\phi(v)$?

It was shown by Even, Itai and Shamir [**?**] that this weighted version of the problem is NP-complete, even for a demand graph H consisting of two independent edges. The purpose of this note is to describe a polynomial-time algorithm which solves WDP for graphs embedded on a fixed compact surface. Thus we denote by $\mathcal{S}$-WDP(H) the restricted version of WDP obtained by constraining the input graph G to be embeddable on $\mathcal{S}$.

A *homeomorph* of a graph H is one obtained by replacing edges of H by paths of varying lengths. Evidentally, the question of whether G contains a homeomorph of H can be answered by solving $O(|V(G)|^{|V(H)|})$ instances of DP - one for each possible mapping of $V(H)$ onto a terminal set in $V(G)$. In the same way, a polytime algorithm for $\mathcal{S}$-WDP implies that there is a polytime algorithm which given a graph G and a multiple $H[\mathbf{d}]$ of H, determines if G contains a subgraph homeomorphic to $H[\mathbf{d}]$. This in turn can be used as follows to find such a subgraph with a maximum number of edges $\sum d_e$. By disjointness of paths, each d_e is clearly bounded by $|E(G)|$. Thus, one need only check homeomorphs of $H[\mathbf{d}]$ for each of these $O(n^{|E(H)|})$ possible $E(H)$-tuples $\mathbf{d}$. Note that this involves checking also *proper* subgraphs of H, that is, we allow some $d_e = 0$. This reasoning now shows that the following problem is tractable.

Maximum H-Multiple Homeomorph (MAX-HOM(H))

INSTANCE: A graph G and weights $\mathbf{w} \in \mathbf{Z}^{E(H)}$.

FIND: Find a maximum weight multiple of H occuring in G as a homeomorph, i.e., $F \subseteq G$ such that $F \cong H[\mathbf{d}]$ with $\sum d_e w_e$ is maximised.

The proofs use one main tool: a family of polytime algorithms due to Schrijver [**?**] which solve the following problem for fixed surfaces $\mathcal{S}$.

Disjoint Homotopic Paths on $\mathcal{S}$ ($\mathcal{S}$-DHP)

INSTANCE: A graph G embedded in $\mathcal{S}$, node pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ and curves $C_1, \ldots, C_k$ such that $C_i(0) = x_i$ and $C_i(1) = y_i$ for $i = 1, \ldots, k$.

Find: A collection of internally disjoint paths $P_1, \ldots, P_k$ such that P_i is homotopic to C_i.

[2] To be precise, the paths are also internally disjoint from $\Phi(V(H))$ in case of isolated nodes in $H[\mathbf{d}]$.

We treat the algorithm for $\mathcal{S}$-DHP as a 'black box' and show how to construct a polynomial number of homotopic 'patterns' which are individually given as input to the algorithm.

A Family of Patterns: $\mathcal{F}(n)$

In the remainder, $\mathcal{S}$ is a fixed compact surface and G represents an arbitrary given connected graph embedded on $\mathcal{S}$. We identify G with its embedding. Moreover, H is a fixed demand graph embeddable on $\mathcal{S}$ and $\Phi : V(H) \to V(G)$ denotes a specification of a terminal set for H in G. Since we can always re-embed our given graph G so that the nodes are positioned appropriately, we assume in fact that Φ maps $V(H)$ onto a *fixed* set T of terminals on $\mathcal{S}$. An H-*image* of G is a subgraph F of G, containing the nodes T, for which (i) each node not in T has degree two and (ii) $F - (T - \{u, v\})$ contains a u, v path only if $uv \in E(H)$. Note that this implies that F is homeomorphic to some $H[\mathbf{d}]$. Call paths described by (ii) *routes.*

We often identify a curve C on $\mathcal{S}$ with its image, and we thus speak of its *endpoints* $C(0), C(1)$, and say C is *internally disjoint* from T if $C(x) \notin T$, $\forall x \in (0, 1)$. Two curves C, D with endpoints x, y are *homotopic*, written $C \sim D$, if one can be continuously deformed into the other leaving its endpoints fixed. More precisely, if there exists a continuous function $\Psi : [0, 1] \to \mathcal{C}_{xy}$ such that $\Psi(0) = C$, $\Psi(1) = D$ and $\mathcal{C}_{xy}$ denotes the set of all curves on $\mathcal{S}$ with endpoints x, y. [3] For background in the topology of surfaces, the reader is referred to [**?**], [**?**].

An H-*pattern* is a finite collection $\mathcal{P}$ of internally disjoint simple curves with endpoints in T, (also internally disjoint from T), such that for each $uv \in E(H)$ there is at least one curve in $\mathcal{C}_{\Phi(u)\Phi(v)}$. The definition implies that $\mathcal{P}$ has an associated embedded graph $G[\mathcal{P}] = (T, \mathcal{P})$ i.e., whose edges are embedded as the curves of $\mathcal{P}$. An H-image F is *fully homotopic* to $G[\mathcal{P}]$ if there is a $1 - 1$ map λ between $\mathcal{P}$ and the curves associated with the routes of F such that $\lambda(P) \sim P$ for each $P \in \mathcal{P}$.

We define a polynomially bounded collection $\mathcal{F}(n)$ of H-patterns such that $\mathcal{F}(n)$ contains all possible homotopy types for n-edge homeomorphs of H-multiples embedded on node set T. In addition, this family can be found (and specified appropriately as input to the algorithm of [**?**]) in polynomial-time. More precisely, $\mathcal{F}$ satisfies:

A *basic H-pattern* is one such that no pair of curves with common endpoints bound a disk. Note that any H-pattern can be thought of as being obtained from a basic one by making copies of some of its curves. The key to polynomially bounding $|\mathcal{F}|$ is to give a constant bound for the size of a basic pattern.

[3] The norm on $\mathcal{C}_{xy}$ may be chosen, for example, as the uniform norm.

- for any graph G with n edges and any $\mathbf{d} \in \mathbf{N}^{E(H)}$: G has an H-image homeomorphic to $H[\mathbf{d}]$ if and only if G has an H-image which is fully homotopic to some $\mathcal{P} \in \mathcal{F}(n)$,
- there is a constant m such that $\mathcal{F}(n) = O(n^m)$.

Lemma 1 *There is a function $f(m,n)$ such that $|\mathcal{P}| \leq f(|V(H)|, g)$ for any basic H-pattern, where g denotes the genus of $\mathcal{S}$.*

Proof: Let F be the faces of $G[\mathcal{P}] = (V', E')$ and F_i be the set of faces with boundary size i, $i \geq 2$. By Euler's Formula: $|V'| - |E'| + |F| = g - 2 + \sum_{\omega \in F} g(\omega)$ where $g(\omega)$ denotes the genus of the surface obtained from ω by contracting to a point the part of $\mathcal{S}$ outside ω, i.e., identifying the points not in ω. [4] In any case, we certainly have the sum of $g(\omega)$'s is at most g and so:

(1) $$|V'| - |E'| + |F| \leq 2g - 2.$$

Also we have

$$3|F| - |F_2| = 3|F - F_2| + 2|F_2| \leq \sum_{i \geq 2} i|F_i| = 2|E'|$$

Thus $|F| \leq \frac{2|E'|}{3} + \frac{|F_2|}{3}$ and by basicness, no face of F_2 bounds a disc and so contains some of the "complexity" of $\mathcal{S}$. Thus $g(\omega) \geq 1$ for each $\omega \in F_2$ and so $|F_2| \leq g$. Hence $|F| \leq \frac{2|E|+g}{3}$ and plugging this back into (**??**) produces:

$$|E'| \leq 3|V'| + 7g - 6.$$

This completes the proof since $|\mathcal{P}| = |E'|$, $|V'| = |V(H)|$ and so $f(H,g) = 3|V(H)| + 7g - 6$ is the desired function. □

This in hand, we now give a polynomial bound on a total collection $\mathcal{F}$. As already noted, any H-pattern on terminals T is obtained from a basic H-pattern $\mathcal{P} = \{C_1, \ldots, C_{|\mathcal{P}|}\}$ by making d_i (strictly positive) copies of curve C_i, $i = 1, \ldots, |\mathcal{P}|$. Denote this collection by $\mathcal{P}[\mathbf{d}]$. If G had an H-image homotopic to $G[\mathcal{P}]$, then by disjointness $\sum_i d_i \leq |E(G)|$ and so the number of available values for each d_i is certainly bounded by $|E(G)|$. Hence the number of feasible m-tuples is at most $|E(G)|^{|\mathcal{P}|}$.

[4] This may create a pinched surface if the boundary of ω is not a simple cycle and if the boundary is a tree, this results in the surface $\mathcal{S}$ again.

Such m-tuples must be checked for each basic pattern, so the total number of patterns investigated is bounded by:

$$\sum_{\mathcal{P} \in \mathcal{B}} n^{|\mathcal{P}|}.$$

where $\mathcal{B}$ denotes the set of all basic H'-patterns where H' is a subgraph obtained by deleting edges but not nodes of H. By Lemma **??** this number is at most

$$|\mathcal{B}| n^{f(H,g)}. \tag{2}$$

The number of patterns in $\mathcal{B}$, again by the lemma, is at most the number of multigraphs which may be obtained by adding no more than $f(H,g)$ edges on the vertex set T. Since this is at most $\binom{|E(H)|+f(H,g)}{|E(H)|} \leq (|E(H)|+1)^{f(H,g)}$ we have from (**??**):

$$|\mathcal{F}(n)| \leq [(|E(H)|+1)n]^{f(H,g)}, \tag{3}$$

and the right hand side is evidentally a polynomial in n.

We now define the collection with the desired properties (**??**).

Definition 1 *For each* $n = 1, 2, \ldots,$
$\mathcal{F}(n) = \{\mathcal{P}[\mathbf{d}] : \mathcal{P}$ *is a basic* H'*-pattern for some* $H' \subseteq H$, $d_i > 0$ *for each* $i = 1, \ldots |\mathcal{P}|$, *and* $\sum_i d_i \leq |E(G)|\}$.

Constructing $\mathcal{F}(n)$

Now that we have defined $\mathcal{F}$ and shown it to be polynomially bounded, the one point remaining is to verify that the curves in $\mathcal{F}$ can indeed be constructed and specified as input to the algorithm of Schrijver. This algorithm specifies such homotopies as simple non-crossing paths in the embedded graph G. Note that these paths are not even necessarily edge-disjoint but that since the paths are non-crossing, they may be slightly shifted to give a proper (re-)embedding of a graph $G[\mathcal{P}]$.

The problem is thus for a given n, to list the members of $\mathcal{F}(n)$ as such a collection of paths. Clearly, it suffices to find the basic patterns $\mathcal{P}$, since it is straightforward to generate such path collections for each of the 'multiples' $\mathcal{P}[\mathbf{d}]$ contained in $\mathcal{F}(n)$ in polynomial-time. Thus we are left with the problem: for a fixed $H' \subseteq H$, generate all basic H'-patterns in $\mathcal{F}(n)$. To do this we need only have a finite algorithm which generates for each H', all the embeddings of H' in $\mathcal{S}$ with the nodes embedded on some fixed set of terminals. Each such embedding is determined by a set of "rotations" at each node identifying its clockwise adjacencies. For each such collection of rotations we determine whether they give rise to an embedding in $\mathcal{S}$ which is 'basic'. Since this last step can be done in finite time, and the number of rotation colections is at most $\Pi_{v \in V(H')} d(v)!$) we can generate the input to Schrijver's algorithm in polynomial-time.

This now completes the proof of the following.

Theorem 2 *For a fixed compact surface $\mathcal{S}$ and demand graph H, the problems* WDP(H) *and* MAX-HOM(H) *are polynomially solvable when restricted to input graphs embeddable on $\mathcal{S}$.*

The family can be applied to other types of disjoint paths or directed paths problems where solutions must consist of internally disjoint paths. In particular, in [**?**], it was stated that for fixed compact surface $\mathcal{S}$ and fixed k, it could be determined in polynomial-time whether a given graph G embeddable on $\mathcal{S}$ contained k induced-disjoint paths between two specified nodes s, t. (A pair of induced internally disjoint $s-t$ paths P_1, P_2 is *induced-disjoint* if there is no edge joining an internal node of P_1 to one of P_2, or equivalentally, if $P_1 \cup P_2$ forms a chordless circuit.) The algorithm for this problem checks all possible homotopy types (patterns) for the k induced-disjoint paths. The number of these is easily polynomially bounded, but the arguments in the present paper imply that we need not fix k.

Theorem 3 *For a fixed surface $\mathcal{S}$, there is a polytime algorithm which given a graph G embeddable on $\mathcal{S}$ and two specified nodes s, t, finds a maximum collection of induced-disjoint $s-t$ paths in G.*

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